Mechanics and Mathematics Reviews

Attached are Mechanics and Mathematics Reviews for incoming *National Test Pilot School* (*NTPS*) students. The reviews were copied verbatim from the Empire Test Pilot School joining instructions and are equally applicable for NTPS students. The reference used for the Mathematics Review is Engineering Mathematics by K. A. Stroud, published by MacMillan Press, LTD, London, England. This book is not required for NTPS students, but it provides a good mathematics review for those students who are interested.

1.0 Mechanics

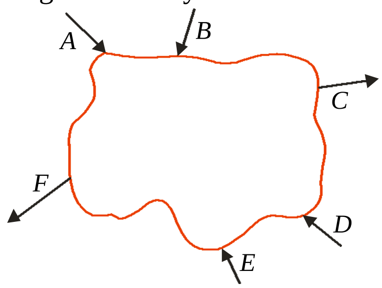
Introduction

1.1 An understanding of basic mechanics is necessary if a study of aircraft in flight is to be undertaken. This chapter briefly reviews the more relevant aspects of simple mechanics.

Equilibrium

1.2 Much of the work of a test pilot is to examine the behavior of an aircraft when disturbed from steady trimmed flight. The trimmed condition is an example of equilibrium: the aircraft is in a state of uniform motion and no out-of-balance forces or moments act to change that state.

1.3 Consider forces acting on a body as shown in Figure 1.1. If we resolve the forces into horizontal and vertical components then one criterion for an equilibrium condition is that the **resultant force** acting on the body is **zero**.



for equilibrium:

Sum of Horizontal Forces to Left = Sum of Horizontal Forces to Right

Sum of 'Up' Forces = Sum of 'Down' Forces

Figure 1.1 Forces in Equilibrium

1.4 This criterion is not the only one, however, as can be seen if we consider Figure 1.2. The horizontal forces balance and the vertical forces balance but the body will **rotate** because the forces do not always act along the same line.

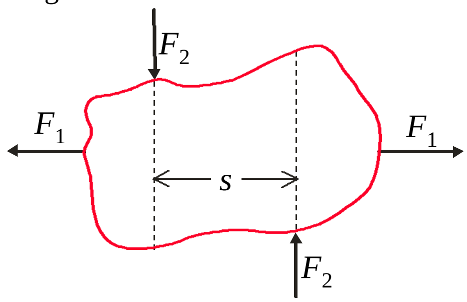
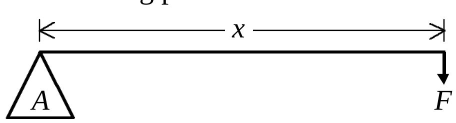


Figure 1.2 Equal Forces: Body not in Equilibrium

1.5 **<u>Moments</u>**. A turning force is called a **moment** and is defined as the product of a force and its perpendicular distance from its turning point.



Moment of F about $A = Fx$

The 'sense' of a moment is important, ie does the body turn clockwise or anti-clockwise under the action of a moment. For equilibrium, therefore, we must not only ensure that the resultant force on the body is zero but also that there is no resultant moment acting on the body.

1.6 **Couples**. Figure 1.2 shows two equal and opposite forces not acting in the same line. This is called a **couple** and the moment of a couple is defined as the produce of one of the forces and the perpendicular distance between them. For example, from Figure 1.2

$$\text{Moment of couple} = F_2 \times s = F_2 s$$

1.7 Summarizing, if a body is in equilibrium then the resultant force on the body is zero and the sum of the moments about any point is also zero.

Note: A body need not be at rest to be in equilibrium to fulfil these conditions. An aircraft in steady, trimmed flight has no resultant force acting, nor any resultant moment: it is in a state of uniform motion, ie moving with constant velocity; and is in equilibrium.

Linear Motion with Constant Acceleration

1.8 Motion with uniform acceleration is important and easy to deal with. If we make our time interval small enough then we can consider any acceleration to be constant. Five quantities are involved in our study of the motion:

Initial speed	u or v_i
Final speed	v or v_f
Acceleration	a or f
Distance covered	s or d
Time taken	t

1.9 The equations governing motion with constant acceleration are easily derived and their derivation can be found in any standard physics text. They are:

$$v = u + at \qquad v^2 = u^2 + 2as \qquad s = ut + 1/2at^2$$

Note: No units are quoted as any system can be used; but the units must be consistent if the equations are to be used.

Newton's Laws of Motion

1.10 The laws are:

First Law "Every body continues in a state of rest or of uniform motion in a straight line unless compelled by external forces to change that state."

Second Law "The rate of change of momentum is proportional to the applied force and takes place in the direction in which the force acts." *NB: The momentum of a body is defined as the product of its mass and velocity*

Third Law "To every action there is an equal and opposite reaction."

1.11 The second law states that the rate of change of momentum is proportional to the applied force. Using appropriate units we can say that

Initial momentum $= mu$

Final momentum $= mv$

Change in momentum $= m(v - u)$

Rate of change: $= \dfrac{m(v-u)}{t}$

$= ma$

(or we could say that rate of change $= \dfrac{d}{dt}(mv) = \dfrac{mdv}{dt} = ma$)

and we have the very important relation for a force, F, that:

$$F = ma$$

ie $\quad$ *Force = Mass × Acceleration*

Circular Motion

1.12 If a body is constrained to move in a curved path, instead of being allowed to continue in a straight line, it will be subject to a radial acceleration. The force producing this acceleration is-called the **centripetal force** and the reaction to this is an inertia force known as the **centrifugal force.**

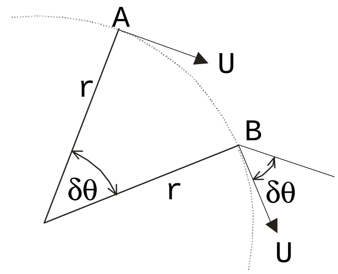


Figure 1.3 - Circular Motion

1.13 Figure 1.3 shows a body moving in a circular path with uniform tangential speed, *U*, thus as the particle moves from *A* to *B* in an increment of time δt:

Length of arc $AB = r\delta\theta$

Speed $U = r\dfrac{\delta\theta}{\delta t}$

Now $\dfrac{\delta\theta}{\delta t} = \omega =$ angular velocity in radians/second

Thus $U = r\omega$

1.14 When the body reaches *B*, its tangential speed is Still *U* but there has been a change in direction, δθ. Thus, the change in velocity towards the center is $U\delta\theta$ and:

Acceleration towards the center $= \dfrac{U\delta\theta}{\delta t} = U\omega$

Using the relation $F = ma$ from Newton's 2nd Law, the centripetal force required to produce this acceleration is

$$F = ma = mr\,\omega^2$$

Centripetal Force $= mr\omega^2$

1.15 This can be written in terms of weight, *W*, and linear velocity, *U*, as

Centripetal Force $= \dfrac{W}{g}\dfrac{U^2}{r}$

16. If we consider an aircraft flying straight and level, its lift, *L*, is equal and opposite to its weight *W*. If, however, it is at the bottom of a pull up from a dive, it is flying in a circular arc. If its velocity is "*U*" and the arc of radius "*r*", an additional lift force must be produced to maintain the aircraft in a circular path.

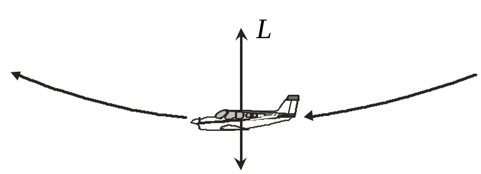


Figure 1.4 - Aircraft in Circular Path

The lift generated must overcome the weight plus the centrifugal force. The total lift is thus:

$$L = W + \frac{WU^2}{gr}$$

If we say that the applied 'g' is n_a: $\frac{U^2}{r} = n_a g$

Then: $L = W(1 + n_a)$

The sum $(1 + n_a)$ is referred to as 'n' the **load factor**

1.17 An accelerometer reads 1 'g' in level flight, ie it is calibrated to read $(1 + n_a)\ g$ ($= ng$) and not $n_a g$

1.18 If we consider the aircraft at the top of a loop then we have the situation as in Figure 1.5 where lift and weight act downwards and the centrifugal force acts 'upwards'.

Thus, $S = \frac{WU^2}{gr} - W = W(n_a - 1)$

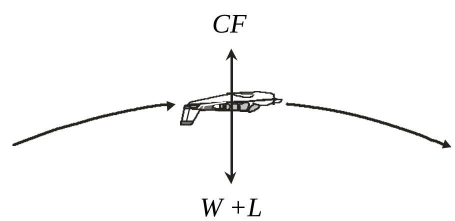


Figure 1.5 Aircraft in Circular Motion

1.19 Figure 1.6 shows the vertical climb and dive positions of the loop where $L = n_a W$ since the lines of action of lift and weight are at 90° to each other.

Note: Aircraft cannot complete a loop of constant radius but the principles involved will be correct at any stages of the 'loop' since at any instant the aircraft is travelling on an arc of a circle of a certain radius.

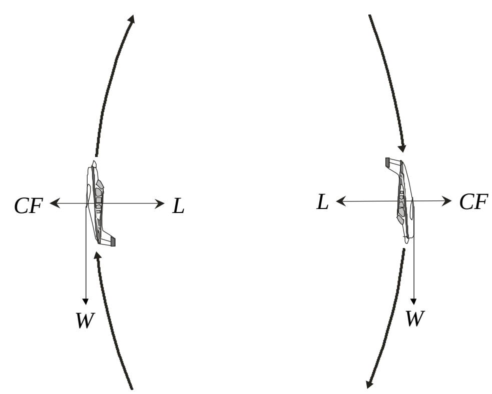


Figure 1.6 Vertical Positions of the Loop

1.20 The following table summarizes the important relations concerning motion of a body in a circular path:

	Linear Motion	Circular (Angular) Motion
Distance	s	$r\theta$
Velocity	U	$r\omega = r\left(\frac{d\theta}{dt}\right)$
Acceleration	a	$r\dot{\omega} = r\left(\frac{d\omega}{dt}\right) = r\left(\frac{d^2\theta}{dt^2}\right)$

Inertia and Moment of Inertia

1.21 Bodies at rest or bodies moving with a constant velocity are in equilibrium. The tendency to continue in the same state of rest or motion is the property called inertia.

1.22 Inertia can only be measured in terms of mass. The mass of the body is a measure of how difficult it is to start or stop.

1.23 When a body is rotating its inertia not only depends on its mass but also on the distance between the axis of rotation and the point at which the mass acts. Consider an aircraft rolling about its longitudinal axis (Figure 1.6).

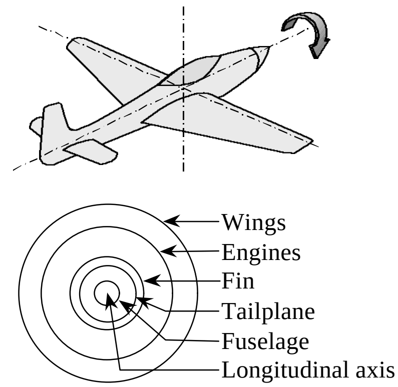


Figure 1.7 Moments of Inertia

The concentric circles represent the paths traced by centers of mass of the parts of the aircraft shown. At any instant the kinetic energy of each part is given by

$$ke = \tfrac{1}{2} m_B v_B^2$$

where $B \equiv$ the wings or engines or tail or fuselage etc

The linear velocity v varies with the distance from the axis hence angular velocity can be substituted and

$$ke = \tfrac{1}{2} m_B r_B^2 \omega^2$$

The total kinetic energy is then the sum of the *ke*'s of all the parts. -~e,

Total $$ke \tfrac{1}{2} \omega^2 \sum m_B r_B^2$$

The quantity $\sum m_B r_B^2$ called the moment of inertia and is represented by I.

1.24 From this it can be seen that the kinetic energy of a rotating body is given by: $\tfrac{1}{2} I\omega^2$

(compare with *ke* of translational motion: $\tfrac{1}{2} mv^2$)

Principles of Angular Momentum

1.25 This is an extension of Newton's Second Law to a rigid body:
"The rate of change of angular momentum of a rigid body rotating about a fixed axis is equal to the moment L about the axis of the external forces acting on the body."

Rate of change of $(I\omega) = L$

$I \times$ rate of change of $(\omega) = L$

ie $$I \times \text{angular acceleration } \frac{d\omega}{dt}\left(\dot{\omega} \text{ or } \frac{d^2\theta}{dt^2}\right) = L$$

ie $$I\dot{\omega} = L \qquad (cf\ ma = F)$$

General Motion of a Rigid Body in Two Dimensions

1.26 It can be shown that in general, the velocity of a body moving in a continuous manner can be considered as compounded of a velocity of translation of a chosen point, for example the center of gravity, and an angular velocity about an axis through the point.

1.27 In fact

The *ke* of a rigid rotor = *ke* of the whole mass assumed concentrated at the *cg* and moving with the velocity of the *cg* + *ke* of rotation of the body about the *cg*

ie $$ke \text{ of a rigid body} = \tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2$$

where v = translational velocity of the *cg*
ω = angular velocity about the *cg*

Summary of Rotational Equations

1.28 The analogy between rotational and translational equations should be noted:

Translational		Rotational	
Mass	M	Moment of Inertia	I
Momentum	$M \times v$	Moment of Momentum	$I\omega$
Force	$M \times a$	Torque	$I \times \dot{\omega}$
Kinetic Energy	$\frac{1}{2} mv^2$	Kinetic Energy	$\frac{1}{2} I\omega^2$

Energy and Power

1.29 **Work**. When a force acts for a distance (ie maintains motion in the direction of the force) it is said to do work. The work done is defined as the product of the force and the distance over which it acts.

$$\text{Work done} = force \times distance$$

1.30 **Energy**. This is defined as the capacity to do work and exists in numerous forms. A body may have energy by virtue of its position (potential energy), its motion (kinetic energy), its temperature (heat energy), and many other forms.

1.31 The **Law of Conservation of Energy** states that "energy can neither be created nor destroyed but merely transformed from one form to another

1.32 a. **Potential Energy**. If a body of mass '*m*' is to be raised through a height '*h'* then we must do work against the force of gravity.

$$\text{Work done} = Mass \times g \times height \quad = mgh$$

At the height '*h*' above ground level the body is said to have **potential energy** stored in it equal to the work done in raising it to that height and available for release on falling.

$\therefore$ PE = mgh

b. **Kinetic Energy.** If a body at rest is accelerated over a distance '*s*' by a force '*F*' until it reaches velocity '*v*' then the work done by the force is $F \times s$ assuming F is constant. If F varies, we can consider its instantaneous value at a point, s, through an increment of length, δs, with change in velocity, δv.

Rate of change of momentum

$$= m\frac{\delta v}{\delta t}$$
$$= m\frac{\delta v}{\delta s}\cdot\frac{\delta s}{\delta t}$$
$$= mv\frac{\delta v}{\delta s}$$
$$= F$$

Work done in moving through δs

$$= F\delta s$$
$$= mv\frac{\delta v}{\delta s}\cdot\delta s$$
$$= mv\delta v$$

∴ Total work done in accelerating from 0 to v is

$$Wd = \int_0^v mvdv$$
$$= \tfrac{1}{2}mv^2$$

This energy is stored in the body by virtue of its motion and is called **kinetic energy**.

$$ke = \tfrac{1}{2}mv^2$$

1.33 **Power**. This is defined as the rate of doing work or the rate of change of energy. If a quantity of energy δE is put into a system in time δt then

$$\text{power} = \frac{\delta E}{\delta t}$$

Revision Sheet No 1

Moments and Equilibrium

1. An aircraft with a tricycle undercarriage is weighed with its datum line horizontal. If the nose wheel weight is 6,250 N and the total main wheel weight 12,500 N and the former is 1 m in front of the datum, and the latter 2 m aft of the datum, where is the *cg*?

(1 m)

2. A tail wheel aircraft is placed on scales and the following data obtained:

Distance of main wheels behind datum	1 m
Distance of tail wheel behind datum	10 m
Weight on each main wheel	9,000 N
Weight on tail wheel	1,000 N

How far is the *cg* of the aircraft behind the datum? (1.47 m)

If a 1000 kg bomb was installed 2 m behind the datum, by how much would the *cg* of the aircraft be altered?

(3 cm farther aft)

3. With tanks empty a certain aircraft has a mass of 10 tons and has a *cg* position 5 cms forward of the aft limit. If the *cg* of the fuel is 25 cms aft of the limit what is the maximum all-up-mass?

(12,000 kg)

If the fuel has a density of 0.7 g/cm^3 and the best specific air range is 1.2 nm/kg, what is the maximum range in still air allowing 50 kg for take-off and climb and 50 kg for descent and landing?

(2,160 nm)

Revision Sheet 2

Motion With Constant Acceleration

1. A cyclist freewheels from rest down an incline of length 100 m with acceleration 2 m/s^2, and continues to freewheel up the opposite slope with retardation 2.5 m/s^2. How many meters are covered on the up slope before it is necessary to pedal?

(80 m)

2. An airplane takes off at a speed of 50 m/s after accelerating uniformly from rest for 200 meters. What is the acceleration, and how long is the airplane in taking off?

(6.25 m/s^2, 8s)

3. A car accelerates uniformly from a speed of 10 m/s to a speed of 30 m/s in travelling 100 m. What is the speed when 50m have been covered?

(22.4 m/s)

4. A train is scheduled to travel a certain portion of its journey at a uniform speed or 30 m/s. If during this part of the journey it was brought to rest with constant retardation in 750 m, and after stopping for 3 minutes is uniformly accelerated in 4 minutes to the scheduled speed of 30 m/s. Find how many seconds the train is then behind its scheduled time.

(325 s)

5. A man on top of a tower of height 100 m, holds his arm over the side of the tower and throws a stone vertically upward with a speed of 10 m/s. Find the height above ground of the highest point reached by the stone, the speed of the stone when it strikes the ground, and the time the stone is in the air. (Take g = 9. 81 m/s^2).

(105.1 m, 45.4 m/s^2, 5.6 s)

Revision Sheet 3

Newton's Laws of Motion

(Take g = 9.81 m/s^2)

1. A coach travelling at 15 m/s pulls up in 50 m. What is the retardation (in m/s^2) if this is uniform? If the coach has a mass of 25 tonnes, what is the retarding force exerted by the brakes?

(2.25 m/s^2, 56.25 kN)

2. A train accelerating uniformly from rest gets up a speed of 15 m/s in 20 seconds. Find how far it has gone when it reaches this speed, and how far it goes in the last second of the acceleration. If the train has a mass of 50 tonnes, the rails are level, and resistance can be neglected, what is the pull of the engine?

(150 m, 14.6 m, 37.5 kN)

3. A man with a mass of 70 kg stands in a lift, which ascends. The motion of the lift consists of three stages, in which:

   a. it accelerates at a rate of 2 m/s^2
   b. it moves at a uniform speed
   c. it is retarded at a rate of 3 m/s^2

   Find the thrust executed by the man on the floor of the lift in each stage of the motion.

(826.7 N, 686.7 N, 476.7 N)

4. The mass of an airplane, including a load of 1000 kg of bombs, is 15 tonnes. If the airplane is in level flight before the bombs are dropped, what will be its vertical acceleration immediately afterwards?

(0.7 m/s^2)

5. A train of mass 200 tonnes starts from rest to climb an incline 1 in 100. The engine exerts a pull of 50 kN, and wind and other resistances total 15 kN. Find how long the train takes to develop a speed of 20 m/s.

(260 s)

Revision Sheet 4

Work Energy and Power

(Take g = 9.81 m/s^2)

1. A car of mass 1 tonne travelling at 25 m/s is brought to rest, by applying the brakes in 100 m. What is the frictional force exerted on the tyres by the road assuming that it is uniform.

(3,125 N)

2. A stone of mass 1 kg is dropped from a height of 2 m into a pond. Find its velocity just before it strikes the pond. The stone begins to sink in the water with a velocity of 2 m/s. What is the energy lost in striking the water?

(6.26/m, 17.62 J)

3. How many Joules of kinetic energy are acquired by a mass of 3 kg in falling freely through a height of 10 m? How much kinetic energy is acquired in falling freely through the next m?

(294.3 J, 294.3 J)

4. A stone is let fall from a height of 50 m above the ground. Find the velocity of the stone as it strikes the ground.
   The ground is sufficiently soft for the stone to penetrate it to a depth of 10 cm. Taking the stone to have a mass of 0.25 kg and the resistance of the ground to penetration to be uniform, find the magnitude of this force of resistance.

(31.3 m/s, 1.225 kN)

5. A car is travelling at 60 m/s on the level, the engine developing 90 KW. Find the resistance to the motion.
   The car, which weighs 1 tonne, then ascends a hill of gradient 1 in 7 at the same speed. What extra power must the engine develop if the other resistances remain unaltered?

(1,500 N, 84 KW)

6. A fire pump delivers 1000 liters of water per minute, the water being initially static and leaving the nozzle at a speed of 20 m/s. The nozzle is 10 m above the source of water supply and the pump is 75% efficient. Taking 1 liter of water to have mass 1 kg, calculate the power required to drive the pump.

(6.63 KW)

Revision Sheet 5
Rotational Motion

Moment of Inertia of a disc about its axis is Mr2/2
Moment of Inertia of a hoop about its axis is Mr2
Take g = 9.81 m/s

1. At the foot of a dive, an airplane is moving at 80 m/s in the arc of a circle of radius 300 m. Find the upthrust on the wings as a fraction of the weight *W* of the airplane.

(3.17 W)

2. A car goes over a hump-backed bridge, the top of which is a circle of radius 20 m. If the car just loses contact with the road at the highest point, find its speed in m/s.

914 m/s)

3. A solid flywheel of radius 1 m is made up of a uniform rim of mass 200 kg supported upon a uniform disc of mass 50 kg. The flywheel is rotating freely at 300 rpm. Calculate the moment of inertia and the kinetic energy.

(225 kgm^2, 11.25 π kJ)

4. A wheel with a diameter of 1.5 m and a mass of 50 kg, which may be regarded as distributed uniformly around the trim, is rolling along a horizontal road at a speed of 10 m/s. Calculate the energy stored in the wheel.

(4,985 J)

5. If it comes to a hill rising 1 in 5 along the road, how far up the incline will it roll before stopping? (In this rolling motion no work is done against friction).

(50 .8 m)

2.0 Mathematics

Introduction

2.1 Post flight analysis of a test program will not usually fall within the domain of a test pilot and he is certainly not expected to be a mathematician. He will, however, need to have an appreciation of the principles underlying his work if he is to be an effective member of a test team.

2.2 Mathematics will only be used on the course where it is relevant in the development of necessary theory. It is therefore essential that some topics be revised prior to this. Some subjects may be familiar, others new; but no attempt will be made to treat the subject rigorously and fuller development will be found in the references. It will be worth your while attempting the exercises if your math is very rusty or if a topic is new in order to check your understanding.

Algebra

2.3 The pre-course material contains a revision of the basic principles of algebra and we need only re-iterate those principles here without laboring the subject.

2.4 **Algebraic Manipulation.**

a. We may only add or subtract like terms:

$$a + a = 2a \qquad 3b - b = 2b$$

but,

$$a + b = a + b$$

b. Any term may be multiplied or divided by another:

$$3a \times 2b = 6ab$$

$$10x \div 5x = 2$$

$$x \div y = \frac{x}{y}$$

2.5. **Brackets**. Bracketed expressions must be treated as an entity:

$$2 \times (3x + 2y) = 6x + 4y$$

$$(12a + 9b) / 3 = 4a + 3b$$

$$(x + y)(x + y) = x^2 + 2xy + y^2$$

2.6. **Factorization**. Is the reverse operation to algebraic multiplication. Common factors can be identified and separated:

$$\tfrac{1}{2}\rho U^2 S(C_D + C_L)$$

or an expression is split into its factors:

$$(x^2 - y^2) = (x - y)(x + y)$$

2.7. **Indices**. The rules are, when **multiplying** two numbers in index form, **add** the indices; when **dividing** two numbers in index form, **subtract** the indices

$$b^3 \times b^2 = b^{3+2} = b^5$$

$$p^5 \div p^2 = p^{5-2} = p^3$$

Note: When any number is raised to the power zero, the answer is 1.

$$x^0 = 1;\ 5^0 = 1;\ b^0 = 1 \text{ etc}$$

Solution of Equations

2.8. There are numerous types of equation including simple linear, simultaneous and quadratic. An example of the solution of each will be worked in detail as an aid to revision:

a. Solve $3(6x + 5) = 51$

(1) **Divide** both sides of the equation by 3:

$$6x + 5 = 17$$

(2) **Subtract** 5 from both sides of the equation:

$$6x = 12$$

(3) **Divide** both sides by the coefficient of x, ie 6:

$$\therefore \quad x = 2$$

b. Solve

$$\left.\begin{aligned} 2x + y &= 5 \\ x - 2y &= 0 \end{aligned}\right\} \quad \begin{aligned} &- (1) \\ &- (2) \end{aligned}$$

(1) **Multiply** equation (1) by 2

$$\left.\begin{aligned} 4x + 2y &= 10 \\ x - 2y &= 0 \end{aligned}\right\} \quad \begin{aligned} &- (3) \\ &- (2) \end{aligned}$$

(2) **Add** equations (3) and (2)

$$5x = 10$$

(3) **Divide** both sides by the coefficient of x, ie 5:

$$\therefore \quad x = 2$$

(4) **Substitute** $x = 2$ in equation (1) or (2):

In eqn (l): $2(2) + y = 5$

$$4 + y = 5$$

Subtract 4 from both sides of the equation

$$y = 1$$

c. Solve $6x^2 + x - 12 = 0$

If this quadratic expression can be factorized, then the equation may be solved by factorization:

$6x^2 + x - 12$ factorizes into $(3x - 4)(2x + 3)$

Thus, the solution of $6x^2 + x - 12 = 0$ becomes

(1) **Factorize** the quadratic expression

$$\therefore \quad (3x - 4)(2x + 3) = 0$$

(2) **Equate** each bracket to zero in turn

$\therefore$ (a) $3x - 4 = 0$

and, solving by the usual method:

$3x = 4$, giving x = 4/3

or (b) $2x + 3 = 0$, giving $2x = -3$, or $x = -3/2$

Quadratic equations may also be solved by the use of the general formula. This method must be used if the expression does not factorize.

d. Solve $2x^2 + x - 5 = 0$

For a quadratic $ax^2 + bx + c = 0$

the solution is given by $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$\therefore$ $$x = \frac{-1 \pm \sqrt{1 - 4(2)(-5)}}{4}$$

$$= \frac{-1 \pm \sqrt{41}}{4} = \frac{-1 \pm 6.403}{4}$$

$\therefore$ either $$x = \frac{-1 + 6.403}{4} = \frac{5.403}{4} = 1.351$$

or $$x = \frac{-1 - 6.403}{4} = \frac{-7.403}{4} = -1.851$$

Logarithms

Definitions

2.9 Consider the expression a^n ie '*a*' multiplied by itself '*n*' times. If we let $A = a^n$ then, provided '*a*' stays constant, the value of '*A*' will depend on the value of '*n*', that is, there is a functional relationship between '*n*' and '*A*'. This relationship is a logarithmic one and $a^n = A$ can be rewritten as:

$n = \log_a A$ (ie n = the logarithm to the base a of A)

2.10 We are familiar with logs to the base 10. For example, tables tell us that $\log_{10} 3 = 0.4771$, which means $10^{0.477}$~ = 3.0. The log of a number to base '*a*' is the power to which '*a*' must be raised to equal the number. For example:

a. $\log_{10} 100 = 2$; because $10^2 = 100$

b. $\log_4 2 = 0.5$ because $\left(4^{0.5} = 4^{\frac{1}{2}} = 2\right)$

Rules Concerning Logarithms

2.11 **Multiplication.** $\log_a xy = \log_a x + \log_a y$

(To multiply two numbers, add their logs and take the antilog).

2.12 **Division** $\log_a (x/y) = \log_a x - \log_a y$

(To divide two numbers, subtract their logs and take the antilog).

2.13 **Raising to a Power**. $\log_a x^n = n \log_a x$

(To raise a number to a power, multiply the log of the number by the power and take the antilog).

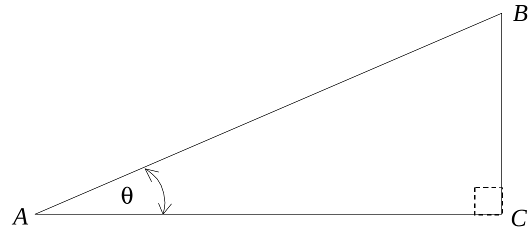
2.14 **Log Logs**. $\log_a (\log_a x^n) = \log_a (n \log_a x)$

$= \log_a n + \log_a (\log_a x)$

Trigonometry

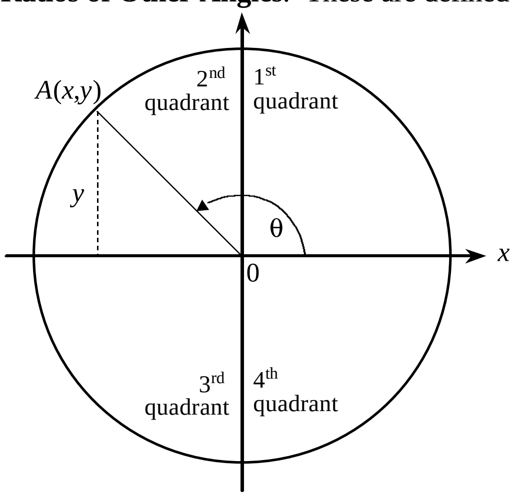
Definitions

2.15 **Ratios of Acute Angles**. The basic trigonometrical ratios of the acute angles are defined in terms of the sides of a right-angled triangle.



$\sin\theta = \dfrac{BC}{AB}$	$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta} = \dfrac{AB}{BC}$
$\cos\theta = \dfrac{AC}{AB}$	$\sec\theta = \dfrac{1}{\cos\theta} = \dfrac{AB}{AC}$
$\tan\theta = \dfrac{BC}{AC}\left(= \dfrac{\sin\theta}{\cos\theta}\right)$	$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{AC}{BC}$

2.16 **Ratios of Other Angles**. These are defined from the following diagram.



Using OX as a reference direction, the angle can be drawn (anti-clockwise positive) making a radial OA where *A* lies on a circle of radius, say, *r*. The point *A* has co-ordinates (*x*, *y*). Then:

$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

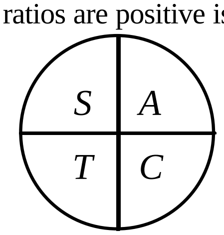
$$\tan\theta = \frac{y}{x}$$

with the signs of *x* and *y* taking their normal positive or negative values depending on which side of the axes they are on and r always being positive

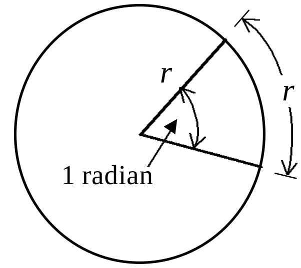
Thus

a. sin θ is positive in the first and second quadrants.
b. cos θ is positive in the first and fourth quadrants.
c. tan θ is positive in the first and third quadrants.

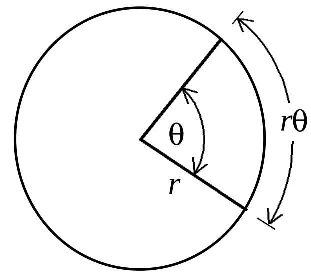
A mnemonic showing which of the ratios are positive is drawn below:



2.17 **Radian Measure**. Practical measurements of angles are made in degrees and minutes, one complete revolution being divided into 360 degrees. This arbitrary number 360 is not suitable for theoretical work and the more basic unit of the radian is used. The radian is defined to be the angle subtended at the center of a circle by an arc equal in length to the radius.



Since the circumference of a circle has a length 2π, one complete revolution is equivalent to an angle of 360° or 2π radians. The conversion factors are therefore:



a. From Degrees to Radians:

$$x \text{ degrees} = \frac{2\pi y}{360} = \frac{\pi y}{180} \text{ radians}$$

b. From Radians to Degrees:

$$y \text{ radians} = \frac{360x}{2\pi} = \frac{180}{\pi}x$$

The length of a circular arc is $r\theta$ where r is the radius of the circle and θ is the angle subtended by the arc (measured in radians).

2.18 **Important Trigonometrical Relationships**. The following trigonometrical relations are stated, without proof, but they will be required in later work:

a. $\sin^2\theta + \cos^2\theta = 1$

b. $\sec^2\theta = 1 + \tan^2\theta$

c. $\sin(A + B) = \sin A \cos B + \sin B \cos A$

d. $\sin(A - B) = \sin A \cos B - \sin B \cos A$

e. $\cos(A + B) = \cos A \cos B - \sin A \sin B$

f. $\cos(A - B) = \cos A \cos B + \sin A \sin B$

g. $\sin 2A = 2 \sin A \cos A$

h. $\cos 2A = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1$
$= 1 - 2\sin^2 A$

2.19 **Graphics Representation of Trigonometric Functions**. By constructing tables of values of sin x and cos x, the graphs of $y = \sin x$ and $y = \cos x$: may be drawn:

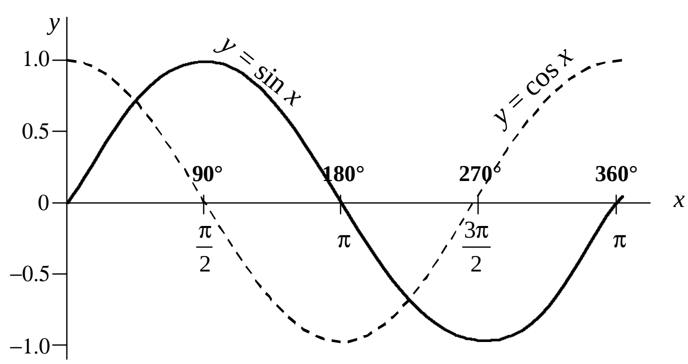


Figure 2.1 Graphs of $y = \sin x$ and $y = \cos x$

If we had drawn the graphs of $y = \sin x$ and $y = \cos x$ over a greater range of values of x, we would have found that the graphs would repeat themselves every 360° (or 2π radians). A function, which repeats itself at regular intervals, is called a periodic function, and the interval between two successive repetitions is called the period of the function. Thus both $\sin x$ and $\cos x$ are periodic functions of x, having periods of 360° (or 2π radians). It can also be seen from the graphs that both sin x and cos x oscillate between a maximum value of +1.0 and a minimum value of -1.0. This maximum/ minimum value of the function is called the amplitude.

2.20 **The Function *A* sin *x***. If each ordinate of the graph of sin x is multiplied by some constant 'A', then the maximum/minimum values will be $\pm A$, but no alteration to the period of the function will occur. Thus the number 'A' determines the amplitude of the function.

2.21. **The Function sin ω*x***. Consider the graph of $y = \sin 2x$ ($\omega = 2$). This will complete one oscillation in 180° so that the period is 180° or π radians. The graph of $y = \sin 3x$ (ie $\omega = 3$) will complete one oscillation in 120° or $\frac{2\pi}{3}$ radians. In general the graph of $y = \sin \omega x$ completes one oscillation in $\frac{360}{\omega}$ degrees or $\frac{2\pi}{\omega}$ radians. Its period is therefore $\frac{2\pi}{\omega}$, that is, the value of ω determines the period. If one oscillation is completed in an interval $\frac{2\pi}{\omega}$, then $\frac{\omega}{2\pi}$ oscillations will occur in an interval of one radian along the x-axis. That is, the **frequency** of the oscillations is $\frac{\omega}{2\pi}$ per radian or ω per revolution (as one revolution = 2π). Thus:

a. Frequency (per radian) $= \dfrac{\omega}{2\pi} = \dfrac{1}{period}$

b. Frequency (per revolution) $= \omega = \dfrac{2\pi}{period} =$ *Angular frequency*

For example:

$V = 240 \sin 100\pi t$ describes an ac signal.

$$\text{Period} = \frac{2\pi}{100\pi} = \frac{1}{50} \text{ second}$$

$$\text{Frequency} = \frac{100\pi}{2\pi} = 50\text{Hz}$$

Angular frequency = 100π rad/sec

Amplitude = 240 volts.

2.22. **Phase Angle**. The graph of $y = \sin\left(x + \frac{\pi}{4}\right)$ has the same period and amplitude as $y = \sin x$ but it is displaced horizontally through π/4. It is said to lead the graph of $y = \sin x$ by π/4 since its maximum/minimum values occur π/4 before those of $y = \sin x$.

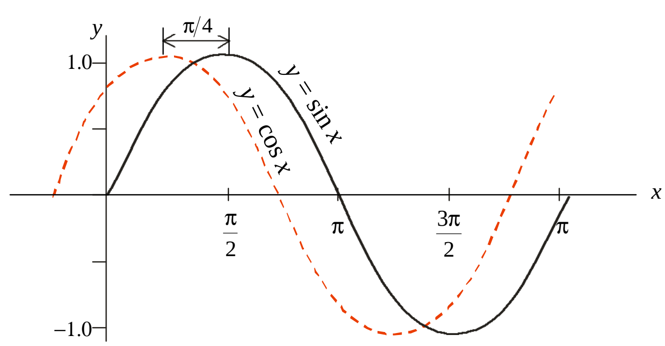


Figure 2.2 Phase Angle

2.23 **Phase Lead/Lag**. In general $y = a \sin(wt + \alpha)$ **leads** $y = a \sin wt$ by α. α is known as the phase lead. If α is negative then $y = a \sin(wt + \alpha)$ **lags** $y = a \sin wt$ and α is known as a phase lag (note this phase lead/lag is an angle as both wt and α are angles).

2.24. **Time Lead/Lag**. Consider $y = a \sin(wt + \alpha)$ and $y = a \sin wt$ again. We have seen that in terms of angles $y = a \sin(wt + \alpha)$ leads $y = a \sin wt$ by the angle α. However in terms of time $y = a \sin(wt + \alpha)$ leads $y = a \sin wt$ by α / w seconds.

2.25. **Phase Difference**. Consider two oscillations represented by the equations:

$$y = a \sin(wt + \alpha) \text{ and } y = b \sin(wt + \beta)$$

The phase of the first at time t is $(wt + \alpha)$ and the phase of the second at the same time is $(wt + \beta)$. The phase difference is the difference between these phases ie. In this case, where we are comparing the phase of the first oscillation with that of the second, the first oscillation is said to **lead** the second if $(\alpha - \beta)$ is positive and to **lag** the second if $(\alpha - \beta)$ is negative. If $\alpha = \beta$ the two oscillations have the same phase at any time and they are said to be in phase. Note that the phase difference is an angle. It is possible to write the phase difference in terms of a time by dividing $(\alpha - \beta)$ by w.

It may be noted that since:

$$a \cos(p\theta + \beta) = a \sin\left(p\theta + \beta + \frac{\pi}{2}\right), \text{ the equation } y = a \cos(p\theta + \beta)$$

represents an oscillation of the same amplitude (a) and period $\left(\frac{2\pi}{p}\right)$ as $y = a \sin(p\theta + \beta)$ but leading it by $\frac{\pi}{2}$.

2.26. **Combination of sin and cos Types of Oscillations**. Consider an oscillation, which is made up of two component oscillations $y = a \sin wx$ and $y = b \sin wx$. Although the amplitudes of these component oscillations are different the frequencies are the same. The final equation of the oscillation may be written as:

$$\begin{aligned} y &= a \sin wx + b \cos wx \\ &\equiv \sqrt{a^2 + b^2} \sin(wx + \delta) \\ &\equiv \sqrt{a^2 + b^2} \cos(wx + \varepsilon) \end{aligned}$$

where: $\tan\delta = \frac{b}{a}$ $\qquad \tan\varepsilon = -\frac{a}{b}$

***The Exponential Function* (e^x)**

Introduction

2.27 In the physical world there are frequent occurrences of physical phenomena in which the rate of change of some quantity is proportional to the quantity itself. This is known as the law of natural growth or decay and some examples are:

a. Newton's Law of Cooling, which states that the rate of decrease at any instant of the excess temperature of a body over its surroundings is proportional to that excess temperature.

b. Radioactive substances decay at a rate, which at any instant is proportional to the quantity of substance present.

This natural growth function is called the **exponential function**, e^x, and it may be defined mathematically in several ways. One of the simplest is to consider the series

$$y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = f(x)$$

Note: ! denotes a "factorial":, $5! = 5 \times 4 \times 3 \times 2 \times 1$

$3! = 3 \times 2 \times 1$

The value of *y* may be found for any value of *x* by substituting that value of *x* into the series.

When: $x = 0$, $y = 1$

$x = \frac{1}{2}, y = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \cdots \cong 1.6487$

$x = 1, y = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots \cong 2.7183$

$x = 2, y = 1 + 2 + 2 + \frac{4}{3} + \frac{2}{3} + \cdots \cong 7.3891$

$x = 3, y = 1 + 3 + \frac{9}{2} + \frac{9}{4} + \frac{27}{3} + \cdots \cong 20.0855$

2.28 Inspection of the results obtained shows that

	1.6487	=	$(2.7183)^{1/2}$
	2.7183	=	$(2.7183)^1$
	7.3891	=	$(2.7183)^2$
	20.0855	=	$(2.7183)^3$
and	1	=	$(2.7183)^0$

2.29 The number 2.7183 . . . obviously has some fundamental relationship with the series and is denoted by the symbol '*e*'.

$e = 2.7183 \ldots$

Hence, when: $x = 0$, $y = e^0$

$x = ½$, $y = e^{½}$

$x = 1$, $y = e^1$

$x = 2$, $y = e^2$

$x = 3$, $y = e^3$

Thus, the series we started with gives:

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = f(x) = e^x$$

This can be proved by more rigorous mathematics, but for our purposes, this is sufficient. Thus:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

and it can be shown that:

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots$$

and

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \frac{a^4x^4}{4!} + \cdots$$

Also, e^x obeys all the normal rules of indices, eg

$$e^a \cdot e^b = e^{a+b} \qquad \frac{e^a}{e_b} = e^{a-b} \qquad \left(e^a\right)^b = e^{ab} \qquad e^{-x} = \frac{1}{e^x}$$

Natural Logarithms

2.30 **Definition**. We have seen, from the definition of logarithms that if $A = a^n$ then $n = \log_a A$. If we use the exponential function, e^x, we can say that if $a = e^x$ then $x = \log_e a$.

Logarithms to the base 'e' are called **natural** or **napierian logs**. Base 10 logs can be easily obtained from natural logs by use of the relation:

$$\begin{aligned}\log_{10} a &= \frac{\log_e a}{\log_e 10} \\ &= \frac{1}{2.3026}\log_e a \\ &= 0.4343\log_e a\end{aligned}$$

Note: '$\log_e a$' is often written as '$\ln a$'; ie ln is an alternative notation to log.

2.31 Graphs of e^x and $\log_e x$. The graph of $y = e^x$ can be plotted using the series of values previously obtained. Other exponential functions are also shown in the figure below:

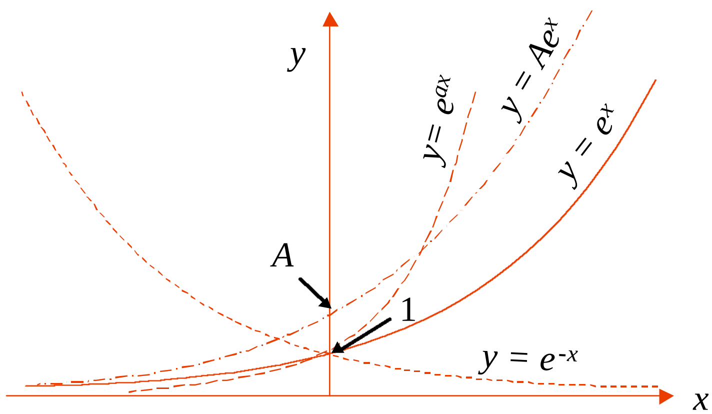


Figure 2.3 Exponential Functions

Points to notice are that:

a. If the index of e^{ax} is positive y will increase as x increases.

b. If the index of e^{ax} is negative y will decrease as x increases.

c. The value of e^{ax} (no matter what value a takes) when $x = 0$ is unity.

d. The graph of $y = A\,e^{ax}$ passes through the point $(0, A)$.

The graph of $y = \log_e x$ is shown below. It is worth noting that as x decreases y tends to minus infinity. It is not possible to obtain values of y for values of x less than zero (ie for negative values of x).

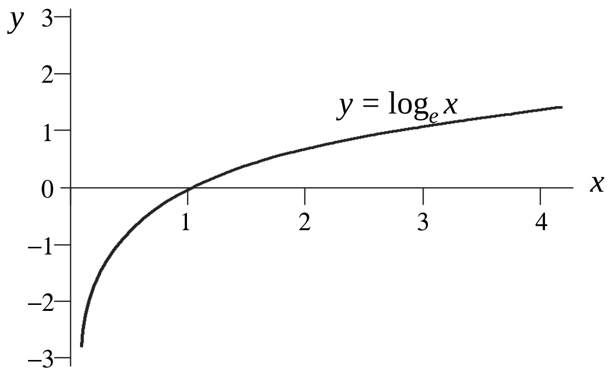


Figure 2.4 Graph of $y = \log_e x$

Relationships between Exponential and Trigonometrical Functions

2.32 Trig functions such as sin x and cos x can be represented as series. These are:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Using series for e^x, we can write out the series for e^{jx} (where $j = \sqrt{-1}$ as used in complex number theory in the pre-course study material):

$$e^{jx} = 1 + jx + \frac{j^2x^2}{2!} + \frac{j^3x^3}{3!} + \frac{j^4x^4}{4!} + \cdots$$
$$e^{jx} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \frac{x^6}{6!} - j\frac{x^7}{7!} + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$
$$= \cos x + j \sin x$$

Thus $$e^{jx} = \cos x + j \sin x$$

Further, had we written out the series for e^{-jx}, we would have obtained:

$$e^{-jx} = \cos x - j \sin x$$

This gives us another relationship since, adding these expressions, we get:

$$e^{jx} + e^{-jx} = (\cos x + j \sin x) + (\cos x - j \sin x)$$
$$= 2 \cos x$$
$$\therefore \cos x = \tfrac{1}{2}\left(e^{jx} + e^{-jx}\right)$$

Also $$e^{jx} - e^{-jx} = (\cos x + j \sin x) - (\cos x - j \sin x)$$
$$= j\, 2 \sin x$$
$$\sin x = \frac{1}{j2}\left(e^{jx} - e^{-jx}\right)$$

Finally $$e^{(a + jb)x} = e^{ax} \cdot e^{jbx}$$

$$e^{(a + jb)x} = e^{ax}(\cos bx + j \sin bx)$$

and similarly $$e^{(a - jb)x} = e^{ax}(\cos bx - j \sin bx)$$

2.33 Summarizing:

a. $e^{jx} = \cos x + j \sin x$

b. $e^{-jx} = \cos x - j \sin x$

c. $\cos x = \frac{1}{2}\left(e^{jx} + e^{-jx}\right)$

d. $\sin x = \frac{1}{j2}\left(e^{jx} - e^{-jx}\right)$

e. $e^{(a \pm jb)x} = e^{ax}(\cos bx \pm j \sin bx)$ -

Complex Number Arithmetic

2.34 As a reminder, first consider some powers of j

a. $j = \sqrt{-1}$

b. $j^2 = \sqrt{-1} \cdot \sqrt{-1} = -1$

c. $j^3 = (j^2)j = -1j = -j$

d $j^4 = (j^2)j^2 = (-1)^2 = 1$

2.35 **Addition and Subtraction**. The rules are simply that we must add (or subtract) the real parts together and the imaginary parts together

$$(4 + j3) + (6 - j) = 10 + j2$$

$$(6 - j2) - (4 + j5) = 2 - j7$$

so, in general, $$(a + jb) + (c + jd) = (a + c) + j(b + d)$$

2.36 **Multiplication**. This is carried out in the same way as you would determine an algebraic product of the form $(3x + 2y)(2x + 5y)$. Eg

$$\begin{aligned}(3 + j6)(7 + j2) &= 21 + j42 + j6 + j^2 . 12 \\ &= 21 + j48 - 12 \\ &= 9 + j48\end{aligned}$$

2.37 **Division**. To perform this operation, we must multiply the numerator and denominator by the **complex conjugate** of the latter in order to convert it to a real number.

$$\begin{aligned}\frac{(7 - j4)}{(4 + j3)} &= \frac{(7 - j4)}{(4 + j3)} \cdot \frac{(4 - j3)}{(4 - j3)} = \frac{28 - j37 - 12}{16 + 9} \\ &= \frac{16 - j37}{25} = \frac{16}{25} - j\frac{37}{25} \\ &= .064 - j1.48\end{aligned}$$

2.38 Equal Complex Numbers. If we know that

$$a + jb = c + jd$$

then $a = c$ and $b = d$

ie the two real parts are equal and the two imaginary parts are equal. ,

Eg If $x + jy = 5 + j2$

then $x = 5$

and $y = 2$

2.39 **Argand Diagrams**. This is a graphical representation of a complex number. They are drawn as vectors on a set of axes with the *x*-axis representing the real part and the *y*-axis the imaginary part of the complex number:

0*A* is the vector representing $(5 + j2)$
0*B* is the vector representing $(-4 + j3)$
0*C* is the vector representing $(-2 - j3)$
0*D* is the vector representing $(3 - j2)$

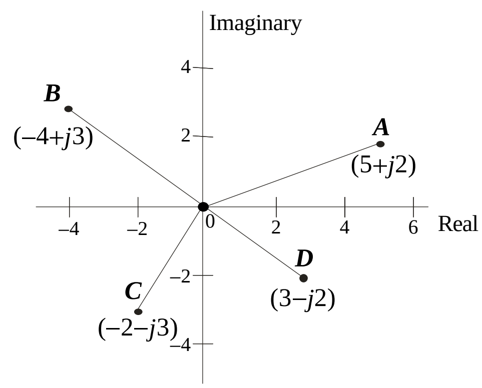


Figure 2.5 Argand Diagram

2.40 **Polar Form of Complex Numbers**. Instead of using the normal cartesian coordinates (x, y) to represent a complex number $(x + jy)$ on a graph we can use the Polar form or (r, θ) form of coordinate. The relation is best explained in this diagram:

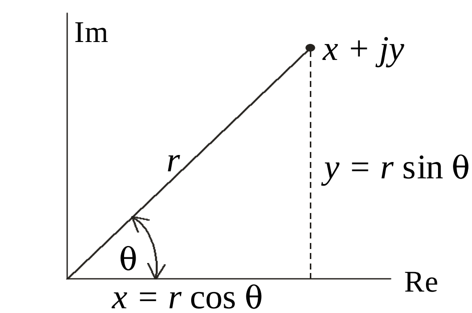


Figure 2.6 Polar Form of Complex Number

2.41 We can see from Figure 2.6 that:

$$r^2 = x^2 + y^2 \qquad \therefore\ r = \sqrt{x^2 + y^2}$$

$$\tan q = \frac{y}{x} \qquad \therefore\ q = \tan^{-1}\frac{y}{x}$$

$$x = r\cos q$$

$$y = r\sin q$$

So, instead of $x + jy$, we can write $(r\cos\theta + jr\sin\theta)$

$$x + jy = r(\cos\theta + j\sin\theta)$$

The form $r(\cos\theta + j\sin\theta)$ is the **polar form** of a complex number and is often shortened to r/θ. *r* is known as the **modulus** of the complex number and θ the **argument**. *r* is **always** positive and θ lies between −180° and +180° (−π rads and +π rads).

2.42 **Exponential Form of a Complex Number**. We have seen previously that

$$e^{j\theta} = (\cos\theta + j\sin\theta)$$

$$\therefore \quad re^{j\theta} = r(\cos\theta + j\sin\theta)$$

Thus, we have written the complex number in exponential form: $re^{j\theta}$

2.43 **Summary**. To summarize, the complex number, 'z', may be written

$z = a + jb$

$z = r(\cos\theta + j\sin\theta)$ polar form

$z = re^{j\theta}$ exponential form

Note: The exponential form is obtained from the polar form, r being the same in both cases, but θ must be in radians in the exponential form.

Differential Calculus

Introduction

2.44 This is an introduction to simple differentiation. The application of differential calculus is relevant to flight test, and the following sections will attempt to outline briefly some of the techniques used. A much fuller study can be found in Stroud.

Function-of-a-Function

2.45 Log x is a function of x, ie the value of log x depends on the value of x. Similarly, $(3x + 2)$ is a function of x, so the value of log $(3x + 2)$ depends on the value of $(3x + 2)$ which itself depends on the value of x.

ie Log $(3x + 2)$ is a function of a function of x

Other examples are $(4x + 5)^3$; $\cos(4x - 3)$; $e^{\sin x}$ etc

2.46 To differentiate such a 'function of a function', consider an example:

Eg Differentiate $y = \sin(2x + 5)$

First, let $u = 2x + 5$ $\therefore \dfrac{du}{dx} = 2$

Now $y = \sin u$

$$\therefore \frac{dy}{du} = \cos u$$

Finally we make use of the relationship

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \cos u \cdot 2$$

but $u = 2x + 5$

$$\therefore \frac{dy}{dx} = 2\cos(2x + 5)$$

2.47 This method can be used for any function of a function. In general terms:

If $y = f(u)$ where $u = f(x)$

then $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

2.48 Should we need to differentiate functions, which are products or quotients of two of the functions then we use the normal rules for differentiating products or quotients together with the rules for 'function of a function'.

Eg If $y = x^2 \sin 5x$

$$\frac{dy}{dx} = x^2 \cdot 5\cos 5x + \sin 5x \cdot 2x$$
$$= x(5x\cos 5x + 2\sin 5x)$$

In general, to differentiate a product

a. Write down the first, differentiate the second, **plus**
b. Write down the second, differentiate the first

Logarithmic Differentiation

2.49 The rules for differentiating products or quotients apply when there are two-factor functions ie uv or $\frac{u}{v}$. When there are more than two functions we use logarithmic differentiation, which makes a problem involving multiplication into one of addition which is easier to handle.

Eg Differentiate $y = x^2 \cdot e^x \sin x$

a. Take logs of both sides In $y = \ln(x^2) + \ln(e^x) + \ln(\sin x)$

b. We know that $\frac{d}{dx}(\ln x) = \frac{1}{x}$, and that if x is replaced by a function of x, say F then

$$\frac{d}{dx}(\ln F) = \frac{1}{F}\frac{dF}{dx}$$

$\therefore$ Differentiating, we get:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2}\cdot 2x + \frac{1}{e^x}\cdot e^x + \frac{1}{\sin x}\cdot \cos x$$
$$= \frac{2}{x} + 1 + \frac{\cos x}{\sin x}$$
$$= \frac{2}{x} + 1 + \cot x$$

$$\therefore \frac{dy}{dx} = y\left(\frac{2}{x} + 1 + \cot x\right)$$

and since $y = x^2 e^x \sin x$

$$\frac{dy}{dx} = x^2 e^x \sin x\left(\frac{2}{x} + 1 + \cot x\right)$$

Applications of Differentiation

Gradient of a Curve

2.50 The basic equation of a straight line is

$$y = mx + c$$

where m = slope, and c = intercept on y axis

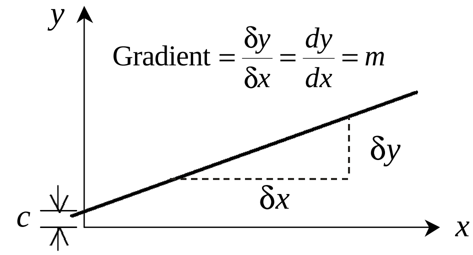


Figure 2.7 Gradient of a Straight Line

A curve, however, will have a continually changing slope or gradient. The gradient of a curve $y = f(x)$ at a point, P, on the curve is given by the slope of the tangent to the curve at the point P. It is also given by the value of $\frac{dy}{dx}$ at the point P, which we can calculate if we know the equation of the curve.

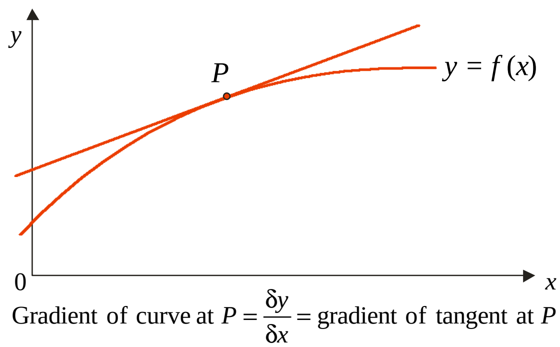


Figure 2.8 Gradient of a Curve

2.51 As an example, consider the curve $y = 2x^2 + 1$. If we wish to find the gradient of the curve at the point (3,2) then we simply have to evaluate $\frac{dy}{dx}$ at that point.

$$y = 2x^2 + 1$$

$$\frac{dy}{dx} = 4x$$

At (3,2), $x = 3$ $$\therefore \frac{dy}{dx} = 12$$

Gradient = 12 at point (3,2)

2.52 **Maxima. Minima and Points of Inflexion**. Consider the graph of some function $y = f(x)$ shown below:

At point A where $x = x_1$, a maximum value of y occurs.

At point B where $x = x_2$, a minimum value of y occurs.

But, at point C we have what looks like a maximum from the left and a minimum from the right. This is called a **point of inflection**.

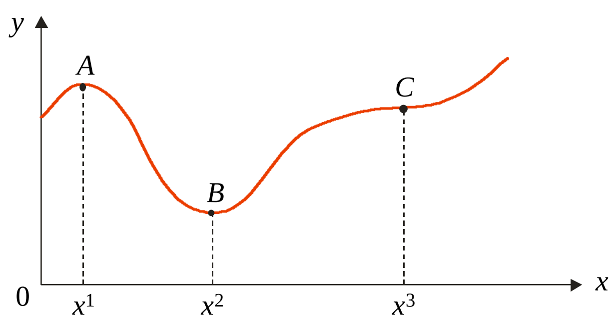


Figure 2.9 Maxima. Minima and Points of Inflexion

2.53 Points *A*, B and *C* are called **turning points** on the graph or **stationary values** of *y*. The important thing about maxima and minima is that the gradient of the curve at either of these points is **zero**. Given the equation of a curve, therefore, we can investigate its shape by studying its maximum and minimum values and any points of inflexion, which may also have a zero gradient.

2.54 It may have occurred to you that if the gradient of the curve is zero we may have any one of the 3 turning points, so we need another criterion to decide which is which.

2.55 Figure 2.10 shows the graph of a curve $y = f(x)$ together with corresponding plots of its gradient as a function of *x* and the second derivative $\frac{d^2y}{dx^2}$ also as a function of *x*.

2.56 Figure 2.10a shows the curve $y = f(x)$, showing a maximum, *A*, a minimum, *C*, and point of inflexion *B*. Following this curve from the *y* axis to the right we see that the slope is at first positive ie upwards to the right, but gradually decreases in steepness until at point *A*, $\frac{dy}{dx} = 0$. Beyond this point the gradient is negative, becoming steeper up to point *B* when it gradually flattens out until, at *C*, it becomes zero again. This produces Figure 10b and a similar process on this figure produces Figure 10c, the variation of the gradient of Figure 10b.

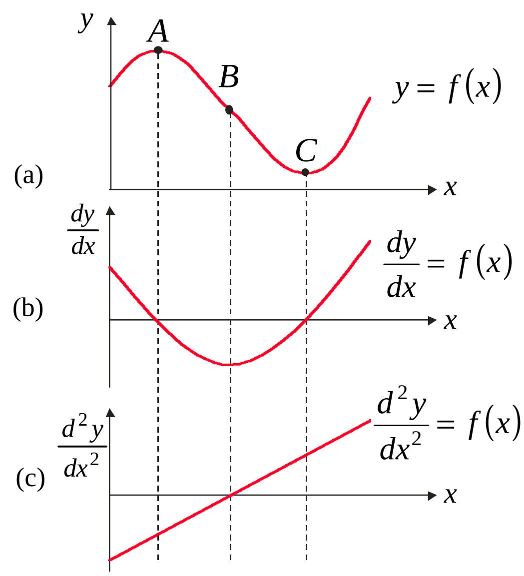


Figure 2.10 Investigation of Maxima Minima and Point of Inflexion

2.57 The slope of the $\frac{dy}{dx}$ curve is given by $\frac{d^2y}{dx^2}$ and it is this which in fact us whether we have a maximum, minimum or point of inflexion. At the point of inflexion, the value of $\frac{d^2y}{dx^2}$ is zero, at a maximum, $\frac{d^2y}{dx^2}$ is negative and at a minimum $\frac{d^2y}{dx^2}$ is positive. All these facts can be verified by consideration of Figure 2.10c.

2.58 To summarize:

For a **maximum** *y*-value: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2}$ is **negative**

For a **minimum** *y*-value: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2}$ is **positive**

For a point of inflexion: $\frac{dy}{dx}$ **may be zero**, $\frac{d^2y}{dx^2} = 0$

2.59 As an example, consider the curve $y = x^3 - 3x$

Differentiating: $\frac{dy}{dx} = 3x^2 - 3$

At turning points, $\frac{dy}{dx} = 0$

$\therefore\ 3x^2 - 3 = 0$

ie $3(x^2 - 1) = 0$

$\therefore\ x = \pm 1$

$\frac{d^2y}{dx^2} = 6x$

When $x = +1$, $\frac{d^2y}{dx^2} = +6$, thus there is a **minimum** at $x = +1$ and $y = -2$

When x = –1, $\frac{d^2y}{dx^2} = -6$, thus a **maximum** occurs at $x = -1$ and $y = 2$

A point of inflexion occurs at $\frac{d^2y}{dx^2} = 0$

ie $6x = 0$

$x = 0$

$\therefore$ Point of inflexion occurs at $x = 0$, $y = 0$.

We can therefore sketch the curve:

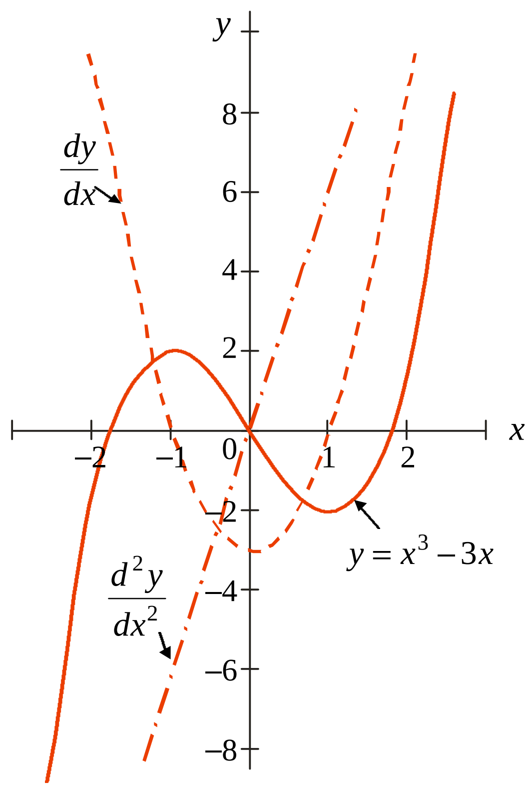


Figure 2.11 Sketch of Curve $y = x^3 - 3x$

Distance. Velocity and Acceleration

2.60 We know, from their definitions, that velocity is the rate of change of distance with time and that acceleration is the rate of change of velocity with time. ie if the distance traveled is '*x*' and time '*t*'

Velocity = $v = \frac{dx}{dt}$

Acceleration = $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

A shorthand notation for differentiation with respect to time is

$$\frac{dx}{dt} = \dot{x} \qquad \frac{d^2x}{dt^2} = \ddot{x}$$

Calculation of Small Changes

2.61 Performance reduction work sometimes requires small adjustments to be made to the results to compensate for slight variations in actual parameters (eg height, speed, temperature etc) from standard values. As an example, let us find what change in engine power, *P*, would result from a small temperature change, *T*.

If we know $\frac{dP}{dT}$ from theory, say, then if *P* and *T* are small, then it may be assumed that:

$$\frac{\delta P}{\delta T} = \frac{dP}{dT}$$

$$\therefore \delta P = \frac{dP}{dT} \cdot \delta T$$

2.62 Eg The periodic time, *T*, of a pendulum is given by:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

If the length, *l*, increases by 1%, find the percentage change in *T*.

If $$T = 2\pi\sqrt{\frac{l}{g}}$$

$$\frac{dT}{dl} = \frac{2\pi}{\sqrt{g}} \cdot \frac{1}{2} l^{-\frac{1}{2}}$$

$$= \frac{\pi}{l} \cdot \sqrt{\frac{l}{g}}$$

$$= \frac{T}{2l}$$

Now $$\frac{\delta T}{\delta l} = \frac{dT}{dl}$$

$$\delta T = \frac{dT}{dl} \cdot \delta l$$

$\therefore$ $$= \frac{T}{2l} \cdot \delta l$$

or $$\frac{\delta T}{T} = \frac{1}{2}\frac{\delta l}{l}$$

But if the increase in length is 1 % $$\therefore \frac{\delta l}{l} = 1\%$$

$$\therefore \frac{\delta T}{T} = \frac{1}{2}\%$$

ie *T* increases by $\frac{1}{2}$%

Note: This technique can be employed in the investigation of the accumulation of errors, which will be covered later in the course when we look at "Error Analysis".

Partial Differentiation

2.63 Differentiation so far has only been used in relationships with one independent variable. If an expression contains two or more independent variables then we have to consider changes in these variables separately. For instance, the pressure, 'p' of a gas depends on its density, 'ρ', we first consider what happens if T changes while remains constant and then look at changes in ρ with T constant.

2.64 The actual relationship is $p = \rho RT$

where R is the gas constant.

If we wish to look at changes in p With respect to T then we must work out $\frac{dp}{dT}$ with ρ constant

ie $\left(\frac{dp}{dT}\right)$ρ constant

This is written as $\frac{\partial p}{\partial T}$ and is known as a **partial derivative**. It may sometimes be written as $\left(\frac{\partial p}{\partial T}\right)_\rho$

Differentiation is now performed as usual, simply assuming all variables constant except the one with respect to which we are differentiating:

ie $$\frac{\partial p}{\partial T} = \left(\frac{\partial p}{\partial T}\right)_\rho = \rho R$$

and $$\frac{\partial p}{\partial \rho} = \left(\frac{\partial p}{\partial \rho}\right)_T = RT$$

2.65 As another example, consider the volume of a cylinder:

The volume of the cylinder V is given by

$$V = \pi r^2 h$$

and is dependent on both radius, r, and height, h. We can find ∂V/∂r and ∂V/∂h

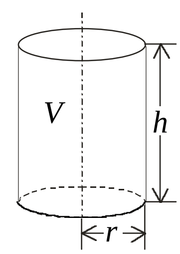


Figure 2.12 The Volume of a Cylinder

If, however, we change r and h simultaneously it can be shown (see Stroud) that the change in volume ∂V is given by :

$$\delta V = \frac{\partial V}{\partial r}\delta r + \frac{\partial V}{\partial h}\delta h$$

2.66 We could extend this result to any number of variables, eg if

$$z = f(x, y, w)$$

then $$\delta z = \frac{\partial z}{\partial x}\cdot\delta x + \frac{\partial z}{\partial y}\cdot\delta y + \frac{\partial z}{\partial w}\cdot\delta w$$

and we can also adapt the idea to include rates-of-change problems. If we divide both sides of the above expression by t, then:

$$\frac{\delta z}{\delta t} = \frac{\partial z}{\partial x}\cdot\frac{\delta x}{\delta t} + \frac{\partial z}{\partial y}\cdot\frac{\delta y}{\delta t} + \frac{\partial z}{\partial w}\cdot\frac{\delta w}{\delta t}$$

In the limit, as $\delta t \to 0$:

$$\frac{\delta z}{\delta t} \to \frac{dz}{dt}; \frac{\delta x}{\delta t} \to \frac{dx}{dt}; \frac{\delta y}{\delta t} \to \frac{dy}{dt}, \text{ and } \frac{\delta w}{\delta t} \to \frac{dw}{dt}$$

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial z}{\partial w}\cdot\frac{dw}{dt}$$

2.67 Eg. The radius of a cylinder increases at the rate of 0.2 cm/sec while the height decreases at the rate of 0.5 cm/sec. Find the rate at which the volume is changing at the instant when r = 8 cm and h = 12 cm.

We know $V = \pi r^2 h$

$$\delta V = \frac{\partial V}{\partial r}\cdot\delta r + \frac{\partial V}{\partial h}\cdot\delta h$$

and $$\frac{dV}{dt} = \frac{\partial V}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial V}{\partial h}\cdot\frac{dh}{dt}$$

Now, $$\frac{\partial V}{\partial r} = 2\pi rh$$

$$\frac{\partial V}{\partial h} = \pi r^2$$

At the instant we are considering, $r = 8$, $h = 12$ and $\frac{dr}{dt} = 0.2$, $\frac{dh}{dt} = -0.5$

$\therefore$ substituting: $$\frac{dV}{dt} = 2\pi(8)(12)\times(0.2) + \pi(8)^2\times(-0.5)$$

$$= 20.1 \text{ cm}^3\text{/sec}$$

Integration

Introduction

2.68 Performance analysis often involves calculation of, for instance, the area under a curve. In such cases, integration can be a great help. We can consider integral calculus to be the reverse of differential calculus, ie if we know the differential coefficient of a function, can we work backwards to find that function?

Eg $$\frac{d}{dx}\left(x^2 + 2\right) = 2x$$

ie the differential coefficient of (x^2 + 2) is 2x. Thus, working backwards we could say that the integral of 2x with respect to x is (x^2 + 2)

The integral sign is $\int$, and, as we are integrating with respect to x, we could write

$$\int 2x\cdot dx = x^2 + 2$$

2.69 But had the function differentiated been $x^2 + 3$ or $x^2 + 4$ or x^2 + (any number), the differential coefficient would always appear simply as $2x$, so without knowing the value of the constant we must in general say that

$$\int 2x \cdot dx = x^2 + K$$

where K is known as the 'constant of integration'. It must **always** be included.

2.70 Continuing the idea of working backwards, every differential coefficient, when written in reverse, gives us an integral.

Eg $$\frac{d}{dx}(\sin x) = \cos x$$

$\therefore$ $$\int \cos x \cdot dx = \sin x + K$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\int e^x \cdot dx = e^x + K$$

2.71 There follows a short list of differential coefficients together with their basic integrals:

1. $\frac{d}{dx}(x^n) = nx^{n-1}$	$\therefore \int x^n \cdot dx = \frac{x^{n-1}}{n+1} + K$ (provided $n \neq -1$)
2. $\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\therefore \int \frac{1}{x} \cdot dx = \ln x + K$
3. $\frac{d}{dx}(e^x) = e^x$	$\therefore \int e^x \cdot dx = e^x + K$
4. $\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\therefore \int e^{ax} \cdot dx = \frac{e^{ax}}{a} + K$
5. $\frac{d}{dx}(\sin x) = \cos x$	$\therefore \int \cos x \cdot dx = \sin x + K$
6. $\frac{d}{dx}(\cos x) = -\sin x$	$\therefore \int \sin x \cdot dx = -\cos x + K$
7. $\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 \cdot dx = \tan x + K$

Table 2.1 List of Standard Differentials and Integrals

2.72. Examples of integration:

1. $\int x^7 \, dx$

this conforms to the (x^n) standard form in our list

$\therefore$ $$\int x^7 \cdot dx = \frac{x^8}{8} + K$$

2. $\int \sin x \, dx$

   Again, a standard form from the list

   $\int \sin x \, dx = -\cos x + K$

3. $\int e^{5x}$

   This is of the standard e^{ax} form in our list

   $$\therefore \quad \int e^{5x} \cdot dx = \frac{e^{5x}}{5} + K$$

4. $$\int \frac{5}{x} \cdot dx$$

   If we were to re-write this as $5\int \frac{1}{x} \cdot dx$

   then we again find a standard form: $5\int \frac{1}{x} \cdot dx = 5 \ln x + K$

   It is worth noting that if at first an integral does not appear to be a standard form, some slight re-arrangement usually changes it to an easily recognizable form. This will become apparent later if not immediately.

Further Integration

2.73 It may be the case that, even after rearranging, an integral does not appear to be of a standard form. The following examples show some other types of 'standard' integral of note:

a. Functions of a Linear Function of *x*.

   Eg $\int (5x - 4)^6 \, dx$

   This is almost of the standard x^n form except that we have $(5x - 4)$ in place of x. To solve this problem we simply make the substitution:

   let $z = (5x - 4)$

   $\therefore$ $\int (5x - 4)^6 \, dx = \int z^6 \, dx$

   and $\int z^6 \cdot dx = \int z^6 \cdot \frac{dx}{dz} \cdot dz$

   From out substitution $z = 5x - 4$

   $$\frac{dz}{dx} = 5$$

   $$\therefore \quad \frac{dx}{dz} = \frac{1}{5}$$

   So our integral becomes:

$$\int z^6 \cdot dx = \int z^6 \cdot \frac{dx}{dz} = \int z^6 \cdot \frac{1}{5} dz$$
$$= \frac{1}{5}\int z^6 dz$$
$$= \frac{1}{5}\frac{z^7}{7} + K$$

Re-substituting for *z* $\int (5x-4)^6 dx = \frac{1}{35}(5x-4)^7 + K$

This substitution method is simple to apply in such cases where *x* is replaced by a linear function of *x* and we note that the normal rules will still apply but we must always **divide by the coefficient of** *x* in the linear function.

ie $$\int x^6 \cdot dx = \frac{x^7}{7} + K$$

$$\int (5x-4)^6 \cdot dx = \frac{(5x-4)^7}{7} \cdot \frac{1}{5} + K = \frac{(5x-4)^7}{35} + K$$

Further examples can be found in Stroud.

b. Integrals of the Form $\int \frac{f'(x)}{f(x)} \cdot dx$ and $\int f(x) \cdot f'(x) \cdot dx$

Note: The differential coefficient of a general function *f*(*x*) is often denoted by *f'*(*x*).

(1) Consider $$\int \frac{3x^2}{x^3-4} dx$$

This is not in our repertoire yet, but notice that if we differentiate $(x^3 - 4)$ we get $3x^2$ ie the differential of the denominator is the numerator in the expression. Thus, make a substitution:

Let $z = x^3 - 4$

Then $\frac{dz}{dx} = 3x^2$ ie $dz = 3x^2 dx$

$\therefore$ $\int \frac{3x^2}{x^3-4} dx$ becomes $\int \frac{dz}{z}$

From our list of standard integrals:

$$\int \frac{dz}{z} = \ln z + K$$

$\therefore$ Substituting for *z*:

$$\int \frac{3x^2}{x^3-4} dx = \ln\left(x^3-4\right) + K$$

We always get this log form of the result when the numerator is the differential coefficient of the denominator. Further examples can be found in STROUD P.461.

(2) Let us now consider $\int \sin x \cdot \cos x\, dx$

In this case, one function (cos *x*) is the differential coefficient of the other.

Make the substitution $z = \sin x$

$\therefore$ $$\frac{dz}{dx} = \cos x$$

ie $dz = \cos x\, dx$

$\therefore$ $\int \sin x \cdot \cos x\, dx$ becomes $\int z\, dz$

and $$\int z\, dz = \frac{z^2}{2} + K$$

thus $$\int \sin x \cos x\, dx = \frac{\sin^2 x}{2} + K$$

For more examples, see Stroud.

c. **Integration by Parts**. For any product where the foregoing rule does not apply we must use another method of integration. This derives from the fact that if '*u*' and '*v*' are functions of *x* then:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides with respect to *x*, we get

$$uv = \int u\frac{dv}{dx} \cdot dx + \int v\frac{du}{dx} \cdot dx$$

Rearranging: $$\int u\frac{dv}{dx} \cdot dx = uv - \int v\frac{du}{dx} \cdot dx$$

or in easier-to-remember terms: $\int u\, dv = uv - \int v\, du$

This gives us a formula for integration of a product, but we must choose one of the factors in our product to be '*u*' and let the other be the **differential coefficient** of same function '*v*'. To find '*v*', therefore, we must integrate this factor.

An example will illustrate this best:

Consider $\int x.\ln x\, dx$

Let $u = \ln x$ and let $dv = x$

thus $$du = \frac{1}{x} \text{ and } v = \int dv = \int x\, dx = \frac{x^2}{2}$$

Substituting in our expression $\int u\, dv = uv - \int v\, du$

$$\int \ln x \cdot x = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}$$
$$= \frac{x^2}{2} \ln x - \int \frac{x}{2}$$
$$= \frac{x^2}{2} \ln x - \frac{x^2}{4} + K$$
$$= \frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) + K$$

Note: Constants of integration are all 'included' in the final constant *K*.

2.74 This is by no means an exhaustive study of integration methods, but will suffice. STROUD gives a far more detailed review should you feel the need to extend your knowledge further.

Integration Applications

Areas Under Curves

2.75 For our purposes, the use of integration to find the area under a curve is most useful. Consider the curve for some function $y = f(x)$ as in Figure 2.13. To find the area under the curve we can divide the area into strips as shown:

2.76. Figure 2.13b shows an enlarged view of the shaded strip. If we approximate the area of the strip to be the area of the rectangle *ABQP* then we have introduced an error with our estimation of the area equal to the area of triangle *PQR*. Thus, approximately:

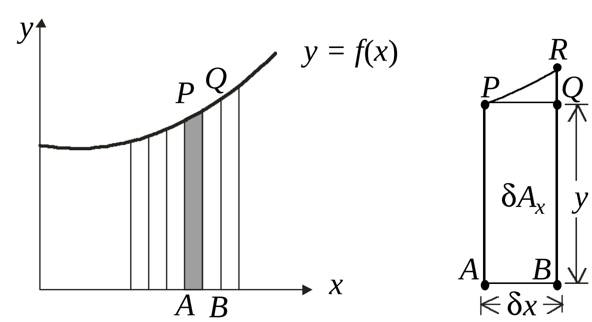


Figure 2.13 Area Under Curve

Total area of strip $= \delta A_x \cong y\,\delta x$

$\therefore$ Approximately $y \cong \dfrac{\delta A_x}{x}$

2.77 As we reduce the width of the strip, the error is reduced until, in the limit, as $\delta x \to 0$.

$$\lim_{dx \to 0} \frac{dA_x}{dx} = \frac{dA_x}{dx} = y$$

and thus is no longer an approximation.

$$\therefore \quad A_x = \int y\,dx = \int f(x)\,dx$$

This gives us the area under the whole curve to the left of point *P*. If *P* is the point (x,y) then we have a method for finding the area under the curve up to any value of *x*. Eg If we let $x = b$ then Figure 2.14a shows the known area. If $x = a$ then Figure 14b shows the known area, and by subtracting one from the other we can find the area under the curve between $x = a$ and $x = b$, Figure 2.14c.

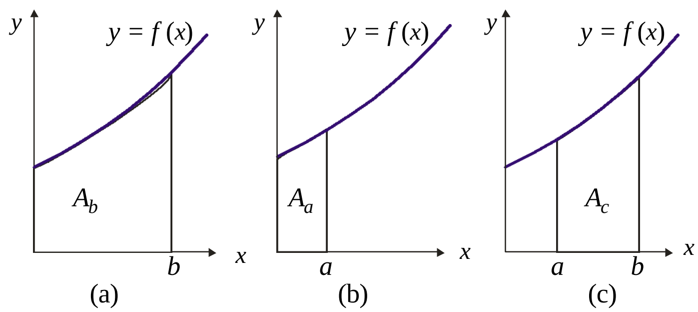


Figure 2.14 Areas Under Curve

The area *A* in Figure 14c is written:

$$A = \int_a^b y \, dx$$

This is known as a **definite integral**, the boundary values '*a*' and '*b*' being the **limits** of the integral. Up to now, all the integrals we have met have been **indefinite,** ie they have no limits.

Note: Definite integrals need no constant of integration.

2.78 Example: Find the area under the curve $y = 2x^2 + 3x + 2$ between $x = 1$ and $x = 4$.

$$A = \int_1^4 \left(2x^2 + 3x + 2\right) dx$$

$$= \left[\frac{2x^3}{3} + \frac{3x^2}{2} + 2x\right]_1^4$$

$$= \left[\frac{2}{3}(4)^3 + \frac{3}{2}(4)^2 + 2(4)\right] - \left[\frac{2}{3}(1)^3 + \frac{3}{2}(1)^2 + 2(1)\right]$$

$$= 74\tfrac{2}{3} - 4\tfrac{1}{6}$$

$$= 70\tfrac{1}{2} \text{ units}$$

Distance, Velocity and Acceleration

2.79 We have seen the relations between these 3 parameters, namely that

$$v = \dot{x}; \;\; a = \ddot{x} = \dot{v}$$

where x = distance, v = velocity, a = acceleration. If we reverse the process, then:

a. Distance traveled between times t_1 and t_2 = $\int_{t_1}^{t_2} v \, dt$

   Note: This is equivalent to the area under the graph of v against t between t_1 and t_2.

b. Velocity between times t_1 and t_2 = $\int_{t_1}^{t_2} a \, dt$

   Note: This is equivalent to the area under the graph of a against t between t_1 and t_2.

Differential Equations

Introduction

2.80 A differential equation is a relationship between an independent variable, x, a dependent variable, y, and one or more differential coefficients of y with respect to x. The real world often produces this type of equation as you will see later,

$$x\frac{dy}{dx}+3xy=0$$

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=x^2$$

$$\frac{d^3y}{dx^3}+4\frac{d^2y}{dx^2}-\frac{dy}{dx}+10=0$$

2.81 The order of a differential equation is given by the highest **derivative** involved in the equation.

$$x\frac{dy}{dx}-y^2=0 \text{ is of the } \textbf{first} \text{ order}$$

$$xy\frac{d^2y}{dx^2}-y^2+2y=x^2 \text{ is of the } \textbf{second} \text{ order}$$

$$\frac{d^3y}{dx^3}-y\frac{dy}{dx}+e^x=0 \text{ is of the } \textbf{third} \text{ order}$$

Solution of Differential Equations

2.82 To 'solve' a de we must find a relationship between x and y for which the equation is true. We do not obtain a numerical answer such as one would expect from an equation (eg the solution of $2x + 1 = 5$ is $x = 2$). We will start with first order de's of which there are several types. An excellent guide to the solution of de's is to be found in STROUD. For this reason, and because most of our work will be concerned with second order equations, only a brief guide to the solution of some types of first order de will be given in this section.

2.83 **Method 1: Direct Integration**. If we can arrange the equation into the form $\frac{dy}{dx} = f(x)$ then we can integrate both sides to solve it:

Example 1 $\quad \frac{dy}{dx}=4x^2+2x+1$

Integrating $\quad \int dy=\int\left(4x^2+2x+1\right)dx$

ie $\quad y=\frac{4}{3}x^3+x^2+x+K$

and this relationship is the solution of the equation.

Example 2 $\quad$ Solve $x\frac{dy}{dx}=5x^3+4$

Dividing both sides by x gives $\quad \frac{dy}{dx}=5x^2+\frac{4}{x}$

and integrating $\quad \int dy=\int\left(5x^2+\frac{4}{x}\right)dx$

ie $\quad y=\frac{5}{3}x^3+4\ln x+K$

2.84 **Method 2: Separating the Variables.** If the equation is of the form $\frac{dy}{dx} f(x, y)$ then we can't solve it by direct integration due to the presence of *y* on the right hand side. If such is the case, we try to rearrange the equation so that all terms in *y* are on one side and all terms in *x* on the other and then integrate. An example will illustrate this:

Example 1 Solve $\frac{dy}{dx} = \frac{x}{y}$

Rearranging: $y\frac{dy}{dx} = x$

Now integrating with respect to *x*: $\int y\frac{dy}{dx} \cdot dx = \int x\, dx$

or $\int y\, dy = \int x\, dx$

Thus $\frac{y^2}{2} = \frac{x^2}{2} + K$

or $y^2 = x^2 + c$

where *K* and *C* are constants.

Example 2 Solve $\frac{dy}{dx} = (1 + x)(1 + y)$

Rearranging: $\frac{1}{(1 + y)} dy = (1 + x)dx$

Integrating $\int \frac{1}{(1 + y)} dy = \int (1 + x)dx$

ie $\ln(1 + y) = x + \frac{x^2}{2} + K$

Linear Differential Equations (First Order)

2.85 First order des are of the general form:

$$\frac{dy}{dx} + Py = Q$$

but we will only concern ourselves with those where $Q = 0$, ie $\frac{dy}{dx} + Py = 0$

STROUD gives a method of solution for this type of equation but we need only to note that the solution is of the exponential type, i.e. of the form:

$$y = Ae^{kx}$$

where *A* and *K*, are constants. These solutions may increase or decrease exponentially depending on the value of *K*: if *K* is positive, *y* will increase or diverge; if *K* is negative then *y* will decrease or converge. We can represent these solutions graphically:

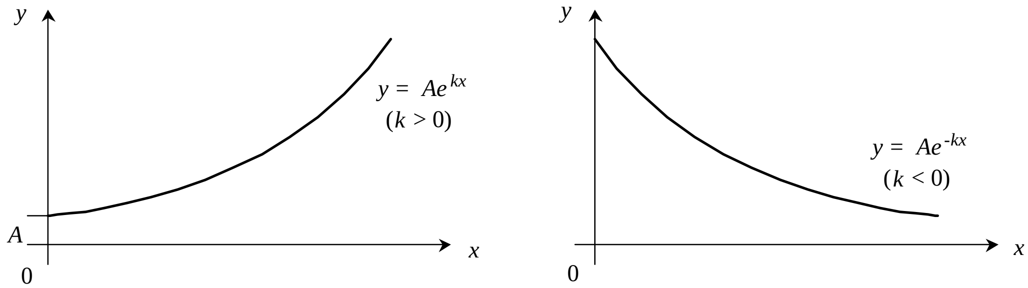


Figure 2.16 Graphs of Exponential Functions

2.86 Without knowing the values of the constants *A* and *K* we say that $y = Ae^{kx}$ is the **general solution** of the equation. This will help with our understanding of the next section on second order de's.

Second Order Differential Equations

2.87 Introduction Many physical systems, when reported mathematically, give rise to second order differential equations of the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

If *a*, *b* and *c* are constants (and not functions of *x* or *y*) then this is an example of a linear **second order de.** Examples of this type of equation arising from the mathematical representation of practical problems are:

1. $m\frac{d^2s}{dt^2} = -ks$ Motion of an undamped spring/mass system.
2. $m\frac{d^2s}{dt^2} + b\frac{ds}{dt} + ks = 0$ Motion of a damped spring/mass system.
3. $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{c} = 0$ Current in an LCR circuit (Inductance, capacitance, resistance)

2.88 All linear differential equations have solutions built up from simple exponential terms of the type $Ae^{\lambda t}$ where *A* and λ are constants. All our work will be with 2nd order equations, usually involving distance, *s*, and time, *t*, so let us consider an example of this:

$$a\frac{d^2s}{dt^2} + b\frac{ds}{dt} + cs = 0$$

Let's guess that a solution will be $s = Ae^{\lambda t}$

where *A* and λ are constants whose value has to be determined. If this is the solution, substituting for *s* in the original equation **must** satisfy the equation.

Thus, if $s = Ae^{\lambda t}$

then $\frac{ds}{dt} = A\lambda e^{\lambda t}$

and $$\frac{d^2s}{dt^2} = A\lambda^2 e^{\lambda t}$$

and substituting into the original equation gives

$$aA\lambda^2 e^{\lambda t} + bA\lambda e^{\lambda t} + cAe^{\lambda t} = 0$$

Dividing by $Ae^{\lambda t}$:

$$a\lambda^2 + b\lambda + c = 0$$

This is a simple quadratic which we can solve to give two values of λ, say λ_1, and λ_2-

ie $$s = Ae^{\lambda_1 t} \quad \text{and} \quad s = Be^{\lambda_2 t}$$

are two solutions of the given equation. It can be demonstrated that if $s = u$ and $s = v$ are two solutions then so is $s = u + v$, thus if

$$s = Ae^{\lambda_1 t} \quad \text{and} \quad s = Be^{\lambda_2 t}$$

are solutions, then so is

$$s = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

It can also be shown that an n^{th} order differential equation has n **arbitrary constants** in its solution, so as there are 2 constants in our solution there is no other solution and this is called the **general solution**.

2.89 The quadratic formed: $$a\lambda^2 + b\lambda + c = 0$$

is called the **auxiliary equation** and is always obtained directly from the equation

$$a\frac{d^2s}{dt^2} + b\frac{ds}{dt} + cs = 0$$

by-substituting λ^2 for $\frac{d^2s}{dt^2}$, λ for $\frac{ds}{dt}$ and 1 (= $\lambda°$) for s

Example 3 Solve $d^2s + 4\frac{ds}{dt} + 9s = 0$

Auxiliary equation: $$\lambda^2 + 4\lambda + 9 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-36}}{2} = \frac{-4 \pm \sqrt{-20}}{2}$$

$$\lambda = \frac{-4 \pm 2j\sqrt{5}}{2} = -2 \pm j\sqrt{5}$$

The solution is thus

$$s = Ae^{\left(-2+j\sqrt{5}\right)t} + Be^{\left(-2-j\sqrt{5}\right)t}$$

This is correct, but looks rather unwieldy, so we will try to re-write it in a neater form:

$$\begin{aligned} s &= Ae^{\left(-2+j\sqrt{5}\right)t} + Be^{\left(-2-j\sqrt{5}\right)t} \\ &= Ae^{-2t}\cdot e^{j\sqrt{5}t} + Be^{-2t}\cdot e^{-j\sqrt{5}t} \\ &= e^{-2t}\left(Ae^{j\sqrt{5}t} + Be^{-j\sqrt{5}t}\right) \end{aligned}$$

Now we know that

$$\begin{aligned} e^{jx} &= \cos x + j\sin x \\ e^{-jx} &= \cos x - j\sin x \\ \text{and that} \quad e^{j\beta x} &= \cos\beta x + j\sin\beta x \\ e^{-j\beta x} &= \cos\beta x - j\sin\beta x \end{aligned}$$

So we can write

$$\begin{aligned} s &= e^{-2t}\left\{A\left(\cos\sqrt{5}t + j\sin\sqrt{5}t\right) + B\left(\cos\sqrt{5}t - j\sin\sqrt{5}t\right)\right\} \\ &= e^{-2t}\left\{(A+B)\cos\sqrt{5}t + j(A-B)\sin\sqrt{5}t\right\} \\ \text{or} \quad s &= e^{-2t}\left\{C\cos\sqrt{5}t + D\sin\sqrt{5}t\right\} \end{aligned}$$

where $C = (A + B)$ and $D = j(A - B)$

Solution of the General Second Order Differential Equation

2.92 **Introduction**. The solution of the general equation

$$a\frac{d^2s}{dt^2} + b\frac{ds}{dt} + cs = 0$$

will be exponential in type, and if we try $s = Ae^{\lambda t}$ as a solution an auxiliary equation is formed:

$$a\lambda^2 + b\lambda + c = 0$$

This equation has two roots given by

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and there are 3 possible sets of roots obtainable:

a. two distinct real roots
b. two identical real roots
c. a pair of complex roots

2.93 **Distinct Real Roots**. If $(b^2 - 4ac)$ is positive, two real roots will be obtained. If these are λ_1, and λ_2 then the general solution will be

$$s = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

2.94 **Identical Real Roots**. If $(b^2 - 4ac)$ is zero, two identical real roots will be obtained. The general solution will be

$$s = e^{\lambda t}(A + Bt)$$

2.95 **Complex Pair of Roots**. If $(b^2 - 4ac)$ is negative, the roots of the auxiliary equation will be complex. If they are $\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \alpha - j\beta$ then the general solution can be written

$$s = e^{\alpha t}(A \cos \beta t + B \sin \beta t)$$

Note: An alternative form of this equation is

$$s = Pe^{\alpha t} \cos(\beta t + \varepsilon)$$

where P and ε are arbitrary constants*. The importance of this is that it allows us to draw a physical significance from the equation. It is a combination of a trigonometric function and an exponential function. The trig part causes the solution to be oscillatory and the exponent part determines whether the amplitude of the oscillation increases or decreases with time. Some important parameters can be determined:

a. Periodic time for the oscillation, $T = \dfrac{2\pi}{\beta}$

b. Time to half amplitude (convergent oscillation). This is the time taken for the amplitude of the oscillation to reach exactly half of its original value and is denoted by $t_{1/2}$.

$$t_{1/2} = \frac{\log_e 2}{-\alpha}$$

c. Time to double amplitude (divergent oscillation)

$$t_2 = \frac{\log_e 2}{\alpha}$$

Note: P and ε are derived from constants A and B by: $P = \sqrt{(A^2 + B^2)}$ and $e = \tan^{-1}\left(\dfrac{-B}{A}\right)$

2.96 Graphical Representation of the Solution. The following diagrams show the variation of the different solutions to the general second order differential equation. The solutions will depend on the roots of the auxiliary equation which we will denote by λ_1 and λ_2.

a. λ_1 and λ_2 both Real and Positive.
Solution is of the form:

$$s = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

in this case, both $Ae^{\lambda_1 t}$ and $Be^{\lambda_2 t}$ represent divergences. The full solution win therefore also be a divergence, the rate of which will depend on the relative magnitudes of its two components.

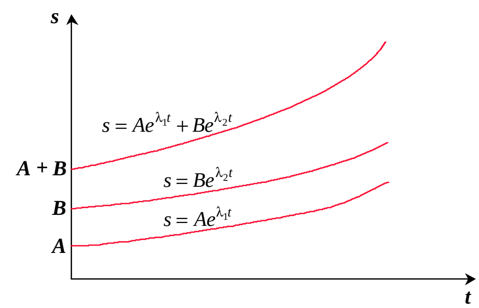


Figure 2.17a Graphical Solution: λ_1 and λ_2 both Real and Positive

b. λ_1 and λ_2 both Real and Negative.
Solution is of the form:

$$s = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

but this time, both components represent damped (convergent) motions and so the full solution will be a damped motion.

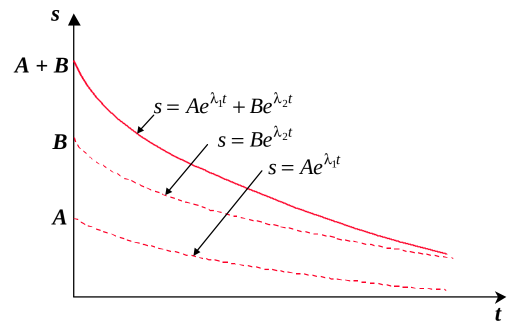


Figure 2.17b Graphical Solution:
λ_1 and λ_2 both Real and Negative

c. λ_1 Real and Negative, λ_2 Real and Positive
Solution is of the form:

$$s = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

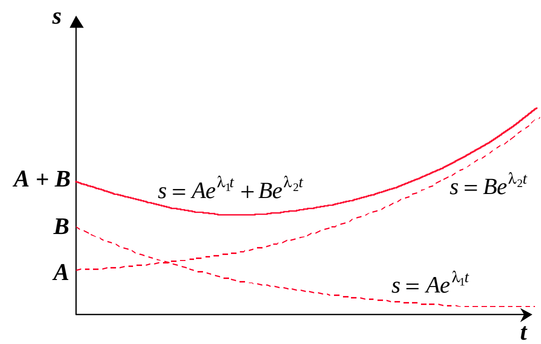


Figure 2.17c Graphical Solution:
λ_1 Positive, λ_2 Negative

d. $\lambda_1 = \lambda_2 = \lambda$
(1) Real and Positive.
Solution is of the form

$$s = e^{\lambda t}(A+Bt)$$

The motion is a divergence as both components represent divergences.

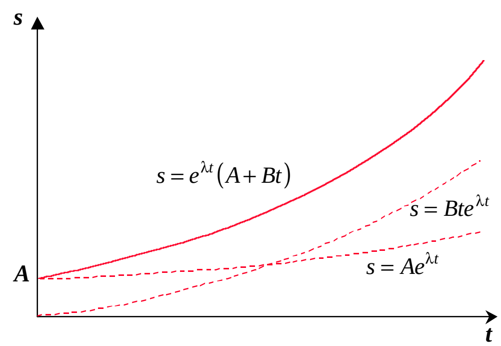


Figure 2.17d(1) Graphical Solution:
$\lambda_1 = \lambda_2 = \lambda$, λ is Real and Positive

(2) Real and Negative
Solution is again, of the form

$$s = e^{\lambda t}(A+Bt)$$

Note: In this particular case, the motion is initially divergent due to the influence of the $Bte^{\lambda t}$ part of the solution

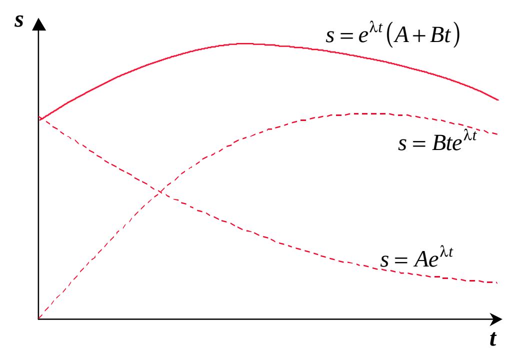


Figure 2.17d(2) Graphical Solution:
$\lambda_1 = \lambda_2 = \lambda$, λ is Real and Negative

e. λ_1 and λ_2 are a Complex Pair
If the roots are

$$\lambda_1 = \alpha + j\beta \qquad \lambda_2 = \alpha - j\beta$$

then we can write the general solution as

$$s = Ae^{\alpha t} \cos(\beta t + \varepsilon)$$

The motion will be oscillatory and convergence or divergence will depend on the sign of the real part of the root ie α.

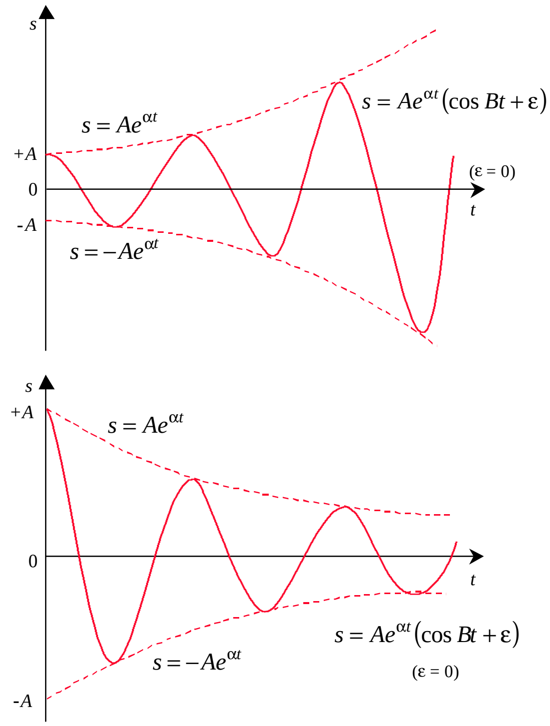


Figure 2.17e Graphical Solution:
λ_1 and λ_2 are a Complex Pair

Calculation of the Arbitrary Constants

2.97 In the preceding sections you will have seen that in solving differential equations we use arbitrary constants as an integral part of the general solution. In this section we will develop a method for calculating these constants.

2.98 The method depends on our being given the "initial conditions" of the problem, ie for a distance-time type problem we may be given some information regarding distance, velocity or acceleration at the start of the motion ie at time t = 0. We **must** have as many independent conditions as there are arbitrary constants in our general solution.

2.99 A worked example will illustrate the method most easily:
Eg Solve the differential equation

$$\frac{d^2s}{dt^2} - \frac{ds}{dt} - 6s = 0$$

subject to the initial conditions that at t = 0, s = 0 and $\dot{s} = 10$.

First, assume a solution of the type $s = Ae^{\lambda t}$, giving auxiliary equation:

$$\lambda^2 - \lambda - 6 = 0$$

$$(\lambda - 3)(\lambda + 2) = 0$$

$$\lambda = 3 \text{ or } \lambda = -2$$

$\therefore$ General solution is

$$s = Ae^{3t} + Be^{-2t} \qquad (1)$$

Now, we know from our initial conditions that at

$$t = 0, s = 0$$

$\therefore$ Substituting in equation

$$0 = Ae^0 + Be^0 = A + B \qquad (2)$$

Differentiating equation (1) we get

$$\dot{s} = \frac{ds}{dt} = 3Ae^{3t} - 2Be^{-2t}$$

and our initial conditions state that at $t = 0$, $\dot{s} = 10$ so substitution gives

$$10 = 3Ae^0 - 2Be^0$$

$$10 = 3A - 2B \qquad (3)$$

We now proceed to solve equations (2) and (3) simultaneously to find A and B.

$$A + B = 0$$

$$3A - 2B = 0$$

Giving $\quad A = 2, B = -2$

Thus, our general solution is now written as

$$s = 2e^{2t} - 2e^{-2t}$$

which is the full solution to the given differential equation subject to the imposed initial conditions.

Determinants

Introduciton

2.100 Stroud covers determinants very well, and hence we will just briefly reiterate the basic principles here.

2.101 Consider a system of equations:

$$a_1X + b_1Y + \partial_1, = 0 \qquad (1)$$

$$a_2X + b_2Y + \partial_2 = 0 \qquad (2)$$

To eliminate Y from these equations multiply eqn (1) by b_1, and eqn (2) by b_2 and then subtract (2) from (1).

$$(a_1 b_2 - a_2 b_1) X = b_1\partial_2 - b_2\partial_1$$

$$X = \frac{b_1\partial_2 - b_2\partial_1}{a_1b_2 - a_2b_1} \qquad (3)$$

From (3) it can be seen that there will only be a non-zero value for X when $a_1 b_2 - a_2 b_1 = 0$.

Definition

2.102 $a_1 b_2 - a_2 b_1$, is therefore an important result as it determines whether or not the system of equations has a non-trivial solution. $a_1 b_2 - a_2 b_1$ is known as a **determinant** and more commonly written as

$$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix}$$

By cross multiplying and subtracting $\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

Solution of Equations

2.103 Eqn (3) can now be written as

$$X = \left| \frac{\begin{matrix} b_1\partial_1 \\ b_2\partial_2 \end{matrix}}{\begin{matrix} a_1b_1 \\ a_2b_2 \end{matrix}} \right|$$

Returning to our original system of equations, (1) and (2), and using the same method as above, it can be shown that:

$$Y = \left| \frac{\begin{matrix} a_1\partial_1 \\ a_2\partial_2 \end{matrix}}{\begin{matrix} a_1b_1 \\ a_2b_1 \end{matrix}} \right|$$

and hence

$$\frac{X}{\begin{vmatrix} b_1\partial_1 \\ b_2\partial_2 \end{vmatrix}} = -\frac{Y}{\begin{vmatrix} a_1\partial_1 \\ a_2\partial_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix}} \qquad (4)$$

2.104 Let us denote

$\begin{vmatrix} b_1\partial_1 \\ b_2\partial_2 \end{vmatrix}$ by Δ_1, $\begin{vmatrix} a_1\partial_1 \\ a_2\partial_2 \end{vmatrix}$ by Δ_2, and $\begin{vmatrix} a_1b_1 \\ a_2b_2 \end{vmatrix}$ by Δ_0.

Eqn (4) becomes:

$$\frac{X}{\Delta_1} = -\frac{Y}{\Delta_2} = \frac{1}{\Delta_0}$$

Example

2.105 Solve by determinants

$$2x + 3y - 14 = 0$$
$$3x - 2y + 5 = 0$$

Now

$$\frac{X}{\Delta_1} = -\frac{Y}{\Delta_2} = \frac{1}{\Delta_0}$$

$$\Delta_0 = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -4 - 9 = -13$$

$$\Delta_1 = \begin{vmatrix} 3 & -14 \\ -2 & 5 \end{vmatrix} = 15 - 28 = 13$$

$$\Delta_2 = \begin{vmatrix} 2 & -14 \\ 3 & 5 \end{vmatrix} = 10 + 42 = 52$$

So that

$$\frac{X}{-13} = -\frac{Y}{52} = \frac{1}{-13}$$
$$X = 1$$
$$Y = 4$$

Matrices

Introduction

2.106 **Definition**. A matrix is a set of or complex numbers arranged in rows and columns to form a rectangular array.

2.107 A matrix having M rows and N columns is called an $M \times N$ (M by N) matrix or a matrix with **order** $M \times N$. A matrix is indicated by writing the array within large square brackets.

$\begin{bmatrix} 4\,2 \\ 1\,9 \\ 8\,7 \end{bmatrix}$ has three rows and two columns and is a 3×2 matrix, order = 3×2

2.108 **Double Sufix Notation**, Each element in a matrix has its own particular location which can be defined by a system of double suffixes. The first suffix denotes the row, the second denotes the column, thus:

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$ with a_{21} indicating the element in the 2nd row lst column.

2.109 M**atrix Notation**. A matrix can be denoted in two main ways:

a. By a single upper case letter

b. By a general element in square brackets

$$\begin{bmatrix} 21 & -49 & 1 \\ 0 & 33 & -50 \end{bmatrix} = C = [c_{ij}]$$

Equal Matrices

2.110 Two matrices are said to be equal if corresponding elements throughout are equal. Therefore, the two matrices must also be of the same order.

$\begin{bmatrix} 7 & 1 \\ -5 & 9 \end{bmatrix} = \begin{bmatrix} \partial_{11} & \partial_{12} \\ \partial_{21} & \partial_{22} \end{bmatrix}$ If $\partial_{11} = 7$, $\partial_{12} = 1$, $\partial_{21} = -5$, $\partial_{22} = 9$

Addition and Subtraction

2.111 To be added or subtracted, 2 matrices must be of the same order and then individual elements are added or subtracted.

$$\begin{bmatrix} 2 & 1 & 5 \\ 3 & 7 & 9 \end{bmatrix} - \begin{bmatrix} 4 & -2 & 1 \\ 3 & 1 & 9 \end{bmatrix} = \begin{bmatrix} -2 & 3 & 4 \\ 0 & 6 & 0 \end{bmatrix}$$

Multiplication of Matrices

2.112 A matrix can be multiplied by either a scalar or another matrix.

a. **Scalar Multiplication**. To multiply a matrix by a single number, a scalar, each individual element of the matrix is multiplied by that factor.

b. **Multiplication of Two Matrices**. Two matrices can only be multiplied together when the number of columns in the first is equal to the number of rows in the second.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 6 & 9 \\ 1 & 5 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 3.4 + 6.1 + 9.2 \\ 1.4 + 5.1 + 3.2 \end{bmatrix} = \begin{bmatrix} 36 \\ 15 \end{bmatrix}$$

It can be seen that multiplying a matrix of order $M \times N$ with a matrix of order $N \times L$ gives a matrix of order $M \times L$.

Transpose of a Matrix

2.113 If the rows and columns of a matrix are interchanged, ie 1st row becomes 1st column etc, the new matrix so formed is called the transpose 'of the original matrix. If A is the original matrix, its transpose is denoted by A^T.

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 90 \\ -25 & 7 \end{bmatrix} \qquad A_T = \begin{bmatrix} 3 & 2 & -25 \\ 4 & 90 & 7 \end{bmatrix}$$

Square Matrix

2.114 A square matrix is a matrix of order $M \times M$

$$\begin{bmatrix} 1 & 9 & 4 \\ 2 & 1 & 3 \\ 8 & 5 & 9 \end{bmatrix} \text{ is a } 3 \times 3 \text{ square matrix}$$

A square matrix [aji] is symmetric if $aji = aji$

$$\begin{bmatrix} 3 & -2 & 1 \\ -2 & 5 & 4 \\ 1 & 4 & 7 \end{bmatrix} \text{ and therefore } A = A_T$$

Unit Matrix

2.115 The unit matrix is one in which the elements on the leading diagonal are all 1, every other element in the matrix is zero.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ The unit matrix is designated by } I$$

It can be shown that $A \cdot I = I \cdot A = A$.

Null Matrix

2.116 The **null** matrix is one whose elements are all zero.

Inverses

2.117 **Inverse of a 2 × 2 Matrix.**

If a matrix $A = \begin{bmatrix} a & b \\ c & \partial \end{bmatrix}$ the Inverse of *A*, written $A^{-1} = \dfrac{1}{DET\ A}\begin{bmatrix} \partial & -b \\ -c & a \end{bmatrix}$

Remember that $DET\ A = \Delta = a\partial - bc$

$$A = \begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix},\ A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -6 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & +1\frac{1}{2} \\ +\frac{1}{2} & -1 \end{bmatrix}$$

If we multiply *A* by $A^{-1} = \begin{bmatrix} 4 & 6 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -\frac{1}{2} & +1\frac{1}{2} \\ +\frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{I}$

In fact it can be shown for any matrix *A* that $AA^{-1} = A^{-1}A = I$

2.118 **Determinants of the Third Order**. A determinant of the third order will contain 3 rows and 3 columns thus:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Each element in the determinant is associated with its **minor**, which is found by omitting the row and column containing the element concerned.

Minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ obtained $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ obtained $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$ obtained $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

2.119 **Evaluation of a Third Order <u>Determinant</u>**. To evaluate a determinant of the third order, we write down each element along the top row, multiply it by its minor and give the terms a plus or minus sign alternatively.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

We already know how to evaluate second order determinants.

Therefore:

$$\begin{vmatrix}1 & 3 & 2\\4 & 2 & 1\\2 & 2 & 3\end{vmatrix} = 1\begin{vmatrix}2 & 1\\2 & 3\end{vmatrix} - 3\begin{vmatrix}4 & 1\\2 & 3\end{vmatrix} + 2\begin{vmatrix}4 & 2\\2 & 2\end{vmatrix} = \quad 4-30+8 \quad = \quad -18$$

2.120 **Determinant of a 3 × 3 Matrix**. The determinant of a 3 × 3 matrix is the determinant having the same elements as those of the matrix.

The determinant of: $\begin{bmatrix}1 & 3 & 2\\4 & 2 & 1\\2 & 2 & 3\end{bmatrix} = \begin{vmatrix}1 & 3 & 2\\4 & 2 & 1\\2 & 2 & 3\end{vmatrix} = -18$

2.121 **Cofactors**. The matrix of cofactors of a matrix A is simply the matrix of all minors of elements in A with the relevant place signs. Place signs are allocated alternatively starting with a + sign for the first element. Therefore for a 3 × 3 matrix place signs would be allocated as follows:

$$\begin{bmatrix}+ & - & +\\- & + & -\\+ & - & +\end{bmatrix}$$

eg

$$\begin{bmatrix}1 & 3 & 2\\4 & 2 & 1\\2 & 2 & 3\end{bmatrix} = A. \quad \text{The matrix of cofactors of } A = C$$

$$C = \begin{bmatrix}+\begin{vmatrix}2 & 1\\2 & 3\end{vmatrix} & -\begin{vmatrix}4 & 1\\2 & 3\end{vmatrix} & +\begin{vmatrix}4 & 2\\2 & 2\end{vmatrix}\\ -\begin{vmatrix}3 & 2\\2 & 3\end{vmatrix} & +\begin{vmatrix}1 & 2\\2 & 3\end{vmatrix} & -\begin{vmatrix}1 & 3\\2 & 2\end{vmatrix}\\ +\begin{vmatrix}3 & 2\\2 & 1\end{vmatrix} & -\begin{vmatrix}1 & 2\\4 & 1\end{vmatrix} & +\begin{vmatrix}1 & 3\\4 & 2\end{vmatrix}\end{bmatrix} = \begin{bmatrix}4 & -10 & 4\\-5 & -1 & 4\\-1 & 7 & -10\end{bmatrix}$$

2.122 **The Adjoint of a Matrix**. The Adjoint of a matrix is the matrix of cofactors transposed. Thus from above:

Adjoint of $A = C^T$, written *ADJ A*

2.123 **Inverse of a Square Matrix**. The adjoint of a matrix is important since it enables us to form the inverse of the matrix. If each element of the adjoint of A is divided by the determinant, providing that determinant is non-zero, the resulting matrix is called the inverse of A.

$$A^{-1} = \frac{1}{|A|} ADJ\ A$$

$$A = \begin{bmatrix}1 & 3 & 2\\4 & 2 & 1\\2 & 2 & 3\end{bmatrix}$$

$$|A| = -18$$

$$\text{Cofactors of } A = C = \begin{bmatrix}4 & -10 & 4\\-5 & -1 & 4\\-1 & 4 & -10\end{bmatrix}$$

$$ADJ\,A = C^T = \begin{bmatrix} 4 & -5 & -1 \\ -10 & -1 & 7 \\ 4 & 4 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-18}\begin{bmatrix} 4 & -5 & -1 \\ -10 & -1 & 7 \\ 4 & 4 & -10 \end{bmatrix}$$

To check this result, we know that $A \cdot A^{-1} = I$

Thus: $$\frac{1}{-18}\begin{bmatrix} 1 & 3 & 2 \\ 4 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} 4 & -5 & -1 \\ -10 & -1 & 7 \\ 4 & 4 & -10 \end{bmatrix} = \frac{1}{-18}\begin{bmatrix} -18 & 0 & 0 \\ 0 & -18 & 0 \\ 0 & 0 & -18 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution of a set of Linear Equations

2.124 Consider any set of linear equations

$$\begin{array}{ccccccccccc} a_{11}X_1 & + & a_{12}X_2 & + & a_{13}X_3 & + & \cdots & + & a_{1n}X_n & = & b_1 \\ a_{21}X_1 & + & a_{22}X_2 & + & a_{23}X_3 & + & \cdots & + & a_{2n}X_n & = & b_2 \\ . & & . & & . & & . & & . & & . \\ . & & . & & . & & . & & . & & . \\ . & & . & & . & & . & & . & & . \\ a_{n1}X_1 & + & a_{n2}X_2 & + & a_{n3}X_3 & + & \cdots & + & a_{nn}X_n & = & b_n \end{array}$$

By simple matrix manipulation this can be written in the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

or, in shorthand $A \cdot X = b$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}; \quad X = \begin{bmatrix} X_1 \\ X_2 \\ . \\ . \\ . \\ X_n \end{bmatrix}; \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

2.125 If we -multiply both sides of the matrix equation by the inverse of A, we have

$$A^{-1} \cdot A \cdot X = A^{-1}b$$

But $$A^{-1}A = I$$

$$\therefore\ IX = A^{-1}b$$

$$X = A^{-1}b$$

2.126 **Example**.

$$X_2 - 4X_2 - 2X_3 = 21$$
$$2X_1 + X_2 + 2X_3 = 3$$
$$3X_1 + 2X_2 - X_3 = -2$$

Equations in matrix form

$$\begin{bmatrix} 1 & -4 & -2 \\ 2 & 1 & 2 \\ 3 & 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 21 \\ 3 \\ -2 \end{bmatrix}$$

$$A \cdot \ X = b$$

$$\therefore \ X = A^{-1}b$$

Cofactors of

$$A = \begin{bmatrix} -5 & 8 & 1 \\ -8 & 5 & -14 \\ -6 & -6 & 9 \end{bmatrix}$$

$$ADJ \ A = \begin{bmatrix} -5 & -8 & -6 \\ 8 & 5 & -6 \\ 1 & -14 & 9 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-39} \begin{bmatrix} -5 & -8 & -6 \\ 8 & 5 & -6 \\ 1 & -14 & 9 \end{bmatrix}$$

$$X = \frac{1}{-39} \begin{bmatrix} -5 & -8 & -6 \\ 8 & 5 & -6 \\ 1 & -14 & 9 \end{bmatrix} \begin{bmatrix} 21 \\ 3 \\ -2 \end{bmatrix} = \frac{1}{-39} \begin{bmatrix} -117 \\ 195 \\ -39 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix}$$

Giving, $X_1 = 3$; $X_2 = -5$; $X_3 = 1$

Eigen Values and Eigen Vectors

Introduction

2.127 Many times in the application of matrices to technological problems involving coupled oscillations and vibrations, equations of the form

$$A \cdot \ X = \lambda X$$

occur, where A is a square matrix, and λ is a number (scalar).

Eigen Values

2.128 If $X = 0$ we clearly have a trivial solution for any value of λ. However, for non-trivial values of X, ie $X \neq 0$ the values of λ are called the **eigen values**, **characteristic values** or **latent roots** of the matrix A.

2.129 In matrix form we have

$$\begin{bmatrix} a_{11} & \cdots & a_n \\ a_{12} & & \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ \\ X_n \end{bmatrix} = \lambda \begin{bmatrix} X_1 \\ X_2 \\ \\ X_n \end{bmatrix}$$

$$\begin{array}{ccccccccc} a_{11}X_1 & + & a_{12}X_2 & + & \cdots & + & a_{1n}X_n & = & \lambda X_1 \\ a_{12}X_1 & + & a_{22}X_2 & + & \cdots & + & a_{2n}X_n & = & \lambda X_2 \\ \cdot & & \cdot & & & & \cdot & & \cdot \\ \cdot & & \cdot & & & & \cdot & & \cdot \\ a_{n1}X_1 & + & a_{n2}X_2 & + & \cdots & + & a_{nn}X_n & = & \lambda X_n \end{array}$$

This simplifies to

$$\begin{array}{ccccccccc} (a_{11}-\lambda)X_1 & + & a_{12}X_2 & + & \cdots & + & a_{1n}X_n & = & 0 \\ a_{21}X_1 & + & (a_{22}-\lambda)X_2 & + & \cdots & + & a_{2n}X_n & = & 0 \\ \cdot & & \cdot & & & & \cdot & & \cdot \\ \cdot & & \cdot & & & & \cdot & & \cdot \\ a_{n1}X_1 & + & a_{n2}X_2 & + & \cdots & + & (a_{nn}-\lambda)X_n & = & 0 \end{array}$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \\ \\ X_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \end{bmatrix}$$

$A \cdot X = \lambda X$ becomes $A \cdot X - \lambda X = 0$ and finally $(A - \lambda I) X = 0$

2.130 For this Set of equations to have a non-trivial solution $|A - \lambda I|$ must be zero. $|A - \lambda I|$ is called the **characteristic determinant** of A and $|A - \lambda I| = 0$ is the **characteristic equation**. Solution of the characteristic equation gives the values of λ, ie the **eigen values** of A.

2.131 Example 1

To find the eigen values of the matrix $A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & 4 & -2 \\ 2 & 10 & -5 \end{bmatrix}$

The characteristic determinant is: $\begin{vmatrix} (2-\lambda) & 3 & -2 \\ 1 & (4-\lambda) & 2- \\ 2 & 10 & (-5-\lambda) \end{vmatrix}$

Expanding this we get the characteristic equation

$$(2 - \lambda)[(4 - \lambda)(-5 - \lambda) + 20] - 3[(-5 - \lambda) + 4] - 2[10 - (4 - \lambda) - 2] = 0$$

$\therefore$ Characteristic equation: $(1 + \lambda)(1 - \lambda)^2 = 0$

$$\therefore \lambda = -1, 1, 1$$

$$\lambda_1 = -1, \ \lambda_2 = 1, \ \lambda_3 = 1$$

Eigen Vectors

2.132 Each eigen value obtained has a corresponding value of X which makes $A \cdot X = \lambda X$ true. This value of X is a column vector called the eigen vector.

2.133 For example, let $A = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}$. Given the equation $A \cdot X = \lambda X$, values of λ can be calculated so that

$\lambda_1 = 1$, $\lambda_2 = 5$.

For $\lambda_1 = 1$ the equation $A \cdot X = \lambda X$ becomes

$$4X_1 + X_2 = X_1$$

$$3X_1 + 2X_2 = X_2$$

either of which gives $X_2 = -3X_1$. Therefore whatsoever value X_1 has, X_2 is always -3 times it. The general eigen vector $\lambda_1 = \begin{bmatrix} K \\ -3K \end{bmatrix}$, therefore there are an infinite number of vectors the simplest of which is $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

2.134 Similarly for $\lambda_2 = 5$

$$4X_1 + X_2 = 5X_1$$
$$3X_1 + 2X_2 = 5X_2$$

Therefore $\lambda_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the simplest eigen vector.

2.135 There are always as many eigen vectors as there are unique eigen values.

Laplace Transforms

2.136 **The Laplace Transform**. The essentials of a physical problem are often contained in a differential equation and certain initial conditions. In the methods of solution previously discussed, it has been necessary first to find the general solution of the differential equation and then to determine the arbitrary constants in this solution from the initial conditions.

2.137 In the case of linear (1^{st} order) differential equations with constant coefficients an alternative method is available using the, so called, **Laplace transform**. The method reduces the technique of solution almost to a "drill' and includes the information given by the initial conditions in the technique of solution.

2.138 A brief description of the method is given below; the method is applicable also to the solution of 2^{nd} order differential equations and its value becomes more apparent as the complexity of the problem under discussion increases:

The Laplace Transform $F(s)$ of a function $f(x)$ is defined by the relation

$$F(s) = \int_0^\infty e^{-st} f(t)dt \qquad (1)$$

it being assumed that the integral on the right exists and that $f(t) = 0$ for all $t < 0$. It is a comparatively simple matter to draw up from (1) a table showing the transforms corresponding to given functions of t.

2.139 For example, if $f(t) = 1$ for all $t > 0$:

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt \\
&= \int_0^\infty 1 \cdot e^{-st} dt \text{ because } f(t) = 1 \text{ for } s > 0 \\
&= \frac{\left[e^{-st}\right]_0^\infty}{-s} = \frac{e^{-\infty} - e^0}{-s} \\
&= \frac{1}{s}
\end{aligned}
$$

2.140 For example, if $f(t) = a \sin bt$ for $t > 0$,

$f(t) = 0$ for $t < 0$,

where a and b are constants;

$$F(s) = a\int_0^\infty (\sin bt)e^{-st} dt$$

Since $e^{jbt} = \cos bt + j \sin bt$

$e^{-jb\,t} = \cos bt - j \sin bt$

we obtain

$$\sin bt = \frac{1}{2j}\left(e^{jbt} - e^{-jbt}\right)$$

Hence

$$
\begin{aligned}
F(s) &= \frac{a}{2j}\int_0^\infty \left(e^{jbt} - e^{-jbt}\right)e^{-st} dt \\
&= \frac{a}{2j} \cdot \frac{1}{s - jb} - \frac{a}{2j} \cdot \frac{1}{s + jb} \\
&= \frac{ab}{s^2 + b^2}
\end{aligned}
$$

2.141 For example, if $f(t) = e^{at}$ where a is a constant

$$
\begin{aligned}
F(s) &= \int_0^\infty e^{at} \cdot e^{-st} dt \\
&= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty \\
&= \frac{1}{(s-a)}
\end{aligned}
$$

2.142 **Worked Example**.

Solve the equation $$\frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y = 0$$

given that $y = 1$, $\frac{dy}{dt} = 0$ when $t = 0$

Multiplying by e^{-st} and integrating with respect to x between 0 and infinity

$$\int_0^\infty e^{-st}\frac{d^2y}{dt^2} dt - 5\int_0^\infty e^{-st}\frac{dy}{dt} dt + 6\int_0^\infty e^{-st} y dt = 0$$

Writing $$y = \int_0^\infty e^{-st} y dt$$

Since $y_0 = 1$, $y_1 = 0$ (suffix denoting the value of the function and its first derivative when $t = 0$)

Then $$\int_0^\infty e^{-st} \frac{dy}{dt} dt = -1 + sy$$

And $$\int_0^\infty e^{-st} \frac{d^2 y}{dt^2} dt = -s + s^2 y$$

Hence the transform can be written

$$-s + s^2 y - 5(-1 + sy) + 6y = 0$$

Thus $$(s^2 - 5s + 6)y = s - 5$$

$$\therefore y = \frac{s-5}{s-5s+6} = \frac{3}{s-2} - \frac{2}{s-3}$$

from para 141, the functions whose transforms are $\frac{1}{s-2}$ and $\frac{1}{s-3}$ are respectively e^{2t} and e^{3t}

Hence $$y = 3e^{2t} - 2e^{3t}$$

2.143 **Worked Example**.
Solve the differential equation

$$\frac{d^2 x}{dt^2} + m^2 x = a \sin nt$$

given that x and $\frac{dx}{dt}$ are both 0 when $t = 0$

$$s^2 x + m^2 x = \frac{an}{s^2 + n^2} \quad \text{(RHS from para 140)}$$

$$x = \frac{an}{(s^2 + m^2)(s^2 + n^2)} \qquad = \frac{a}{m^2 - n^2} \cdot \frac{n}{s^2 + n^2} - \frac{n}{m} \cdot \frac{m}{s^2 + m^2}$$

Therefore $$x = \frac{a}{m^2 - n^2} \sin nt - \frac{n}{m} \sin mt$$

2.144 Using the attached table of Laplace Transform parts we can now move forward to developing system transforms and investigating their response to different forms of input signal.

Laplace Operations

#	Time Domain	Laplace Domain	Description
1	$x(t)$ or x	$X(s)$ or X	Notation
2	$\frac{dx}{dt}$	$sX - X_{0+}$	First derivative
3	$\frac{d^2x}{dt^2}$	$s^2X - s(X)_{0+} - \left(\frac{dx}{dt}\right)_{0+}$	Second derivative
4	$\int_{-\infty}^{t} xdt$	$\frac{1}{s}\left[X + \int_0^{-\infty} xdt\right]$	Integral
5	$\int_0^t xdt$	$\frac{X}{s}$	Integral with constant of integration = 0
6	$\lim_{t \to 0}(x)$	$\lim_{s \to \infty} s(x)$	Initial value theorem
7	$\lim_{t \to \infty}(x)$	$\lim_{s \to 0} s(x)$	Final value theorem (with restrictions)
8	$x + y$	$X + Y$	Addition of transformations
9	kx	kX	Multiplication by a contrast

Note: $L\{x(t)y(t)\}$ is not $X(s)Y(s)$ Multiplication of time functions is not the equivalent of multiplication of Laplace functions.

Inputs

#	Time Domain	Laplace Domain	Description
1	(t)	1	Unit Impulse
2	$a(t)$	a	Impulse *a*
3	1	$\frac{1}{s}$	Unit step
4	a	$\frac{a}{s}$	Step *a*
5	t	$\frac{1}{s^2}$	Unit ramp
6	at	$\frac{a}{s^2}$	Ramp *a*

Additional Laplace Transforms $F(s) = \mathrm{L}f(t) = \int_0^\infty f(t)e^{-st}dt$

#	Laplace Domain $F(s)$	Time Domain $f(t)$	#	Laplace Domain $F(s)$	Time Domain $f(t)$
1	$aF(s)$	$af(t)$	**17**	$\frac{\omega}{s^2+2as+a^2+\omega^2}$	$e^{-at}\sin\omega t$
2	$\frac{a}{s}$	a	**18**	$\frac{s+a}{s^2+2as+a^2+\omega^2}$	$e^{-at}\cos\omega t$
3	$\frac{1}{s}$	1	**19**	$\frac{s+b}{s^2+2as+a^2+\omega^2}$	$\frac{1}{\omega}\left[(b-a)^2+\omega^2\right]^{1/4}e^{-at}\sin(\omega t+\Psi)$ where $\Psi=\tan^{-1}\frac{\omega}{b-a}$
4	$\frac{1}{s^2}$	t	**20**	$\frac{1}{(s+a)(s+a)}$	$\frac{e^{-at}-e^{-bt}}{b-a}$
5	$\frac{n!}{s^n+1}$	t^n	**21**	$\frac{s+a}{(s+b)(s+c)}$	$\frac{(a-b)e^{-bt}-(a-c)e^{-at}}{c-b}$
6	$\frac{1}{s+a}$	e^{-at}	**22**	$\frac{s+a}{s^2+\omega^2}$	$\frac{1}{\omega}\left[a^2+\omega^2\right]^{1/4}\sin\left(\omega^t+\Psi\right)$ where $\Psi=\tan^{-1}\frac{\omega}{a}$
7	$\frac{s}{s+a}$	$f(0) - ae^{-at}$	**23**	$\frac{s+a}{s(s^2+\omega^2)}$	$\frac{1}{\omega^2}\left[a-\left(a^2+\omega^2\right)^{1/4}\right]e^{-at}\cos(\omega t+\Psi)$ where $\Psi=\tan^{-1}\frac{\omega}{a}$
8	$\frac{s+a}{s+b}$	$f(0)+(a-b)e^{-bt}$	**24**	$\frac{1}{s(s+a)}$	$\frac{1-e^{-at}}{a}$
9	$\frac{1}{(s+a)^2}$	te^{-at}	**25**	$\frac{s+a}{s(s+b)}$	$a^b+\left(1-\frac{a}{b}\right)e^{-bt}$
10	$\frac{n!}{(s+a)^{n-1}}$	$t^n e^{-at}$	**26**	$\frac{1}{s(s+a)^2}$	$\frac{1-(1+at)e^{-at}}{a^2}$
11	$\frac{s+a}{(s+b)^2}$	$[(a-b)\,t+1]\,e^{-bt}$	**27**	$\frac{s+a}{s(s+b)^2}$	$\frac{a}{b^2}+\left(\frac{b-a}{b}t-\frac{a}{b^2}\right)e^{-bt}$
12	$\frac{\omega}{s^2+\omega^2}$	$\sin\omega t$	**28**	$\frac{1}{s(s+a)(s+a)}$	$\frac{1}{ab}\left[1+\frac{be^{-at}-ae^{bt}}{(a-b)}\right]$
13	$\frac{s}{s^2+\omega^2}$	$\cos\omega t$	**29**	$\frac{1}{(s^2+\omega^2)^2}$	$\frac{\sin\omega t-\omega t\cos\omega t}{2\omega^2}$
14	$\frac{\omega^2}{s(s^2+\omega^2)}$	$1-\cos\omega t$	**30**	$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t\sin wt}{2w}$
15	$\frac{\omega^2}{s^2(s^2+\omega^2)}$	$\frac{t-\sin\omega t}{\omega}$	**31**	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t\cos\omega t$
16	$\frac{\omega^2}{s^2(s^2+\omega^2)}$	$\frac{t^2}{2}+\frac{\cos\omega t-1}{\omega^2}$			

Assignment 1

1. Express with positive indices instead of root signs

   a. $\sqrt{5}$

   b. $3\sqrt{3}$

   c. $\dfrac{1}{4\sqrt{2}}$

   d. $\sqrt{x^3}$

   e. $\sqrt{4m}$

2. Simplify:

   a. $\left(\dfrac{2}{a}\right)^2 + \left(\dfrac{1}{2a}\right)^3$

   b. $\dfrac{(3x)^4}{3x^4}$

   c. $\dfrac{(xy^2)^3}{(xy^3)^2} \times \left(\dfrac{y}{x}\right)^4$

   d. $y^{\frac{1}{5}} \times y^{\frac{3}{10}}$

   e. $\dfrac{p^{\frac{1}{3}} \times p^{\frac{1}{4}}}{p^{-\frac{1}{12}}}$

3. Solve:

   a. $12x^2 + 29x - 8 = 0$ b. $t^2 = t + 6$ c. $\dfrac{r}{4} = \dfrac{5}{5}$

4. Simplify:

   a. j^3 b. j^5 c. j^{12} d. j^{14}

5. Express in the form $a + jb$
   a. $(4 - j7)\ (2 + j3)$
   b. $(-1 + j)2$
   c. $(5 + j2)(4 - j5)(2+j3)$
   d. $\dfrac{4 + j3}{2 - j}$

6. Express in polar form
   a. $3 + j5$
   b. $-6 + j3$
   c. $-4 - j5$

7. Express in the form $a + jb$
   a. $5(\cos 225° + j \sin 225°)$
   b. $|\underline{1330}$

8. Find the values x and y that satisfy the equation
   $(x + y) + j(x - y)\ 14.8 + j6.2$

9. Differentiate the following with respect to x:
   a. $\tan 2x$
   b. $(5x + 3)6$
   c. $\log_{10}(x^2 - 3x - 1)$
   d. $\ln \cos 3x$
   e. $\sin^3 4x$
   f. $e^{2x} \sin 3x$
   g. $\dfrac{x^4}{(x+1)^2}$
   h. $\dfrac{e^{4x} \sin x}{2x \cos 2x}$

10. Differentiate with respect to x:
    a. $\ln\left[\dfrac{\cos x + \sin x}{\cos x - \sin x}\right]$
    b. $\ln(\sec x + \tan x)$
    c. $\sin^4 x \cos^3 x$

11. Find $\frac{dy}{dx}$ when
    a. $y = \dfrac{x \sin x}{1 + \cos x}$
    b. $y = \ln\left[\dfrac{1 - x^2}{1 + x^2}\right]$

Assignment 2

1. Integrate with respect to x

   a. $\dfrac{3x^3 + 4x^2 + 5}{x^2}$

   b. $(ax + b)(cx + d)$

   c. $(x^2 + 1)^3$

   d. $\left(ax + \dfrac{b}{x}\right)\left(ax - \dfrac{b}{x}\right)$

2. Find the area bounded by the curves $y = 3e^{2x}$ and $y = 3e^{-x}$ and the ordinates at $x = 1$ and $x = 2$.

3. Evaluate the following definite integrals:

   a. $\int_0^{\pi/2} 2\sin^2 x\, dx$

   b. $\int_0^{a} \cos^2(x\text{-}a)\, dx$

   c. $\int_0^{\pi/2} \left(2\cos^2\theta + 3\sin^2\theta\right) d\theta$

   d. $\int_0^{\pi/2} \left(2\cos^2\theta + 3\sin\theta\right) d\theta$

   e. $\int_0^{\pi/3} 2\sin 3x \cos x\, dx$

   f. $\int_0^{\pi/3} \sin x \cos x\, dx$

4. Integrate the following expressions with respect to x:

   a. $(2 - x)^3$

   b. $(1 - x)^{10}$

   c. $(2x - x)^{-3}$

   d. $\dfrac{1}{\sqrt{(5x - 7)}}$

   Evaluate the following definite integrals:

   e. $\int_0^{\pi/6} \sin\left(\dfrac{\pi}{3} - 20\right) d\theta$

   f. $\int_0^{\pi/4} \cos 2t\, dt$

Assignment 3

1. Calculate

a. $\begin{vmatrix} 5 & 6 \\ 7 & 4 \end{vmatrix}$ b. $\begin{vmatrix} 5 & -2 \\ -3 & -4 \end{vmatrix}$ c. $\begin{vmatrix} a & d \\ b & c \end{vmatrix}$ d. $\begin{vmatrix} p & q \\ r & s \end{vmatrix}$

2. Solve by determinants:

a. $4x + 2y + 20 = 0$
$3x + 2y - 2 = 0$

b. $2x - y = 3$
$x + 5y = 2$

c. $x - 3y = -2$
$x + y = 4$

3. If $A = \begin{bmatrix} 7 & 2 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 6 \\ 5 & 8 \end{bmatrix}$ determine

a. $A + B$

b. $A - B$

c. $A \cdot B$

d. $B \cdot A$

4. If

$A = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}$ $J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

express

a. $A \cdot B$

b. $B \cdot C$

c. $A \cdot C$

d. A^2

in terms of other matrices